

SEMICLASSICAL ANALYSIS OF LOW AND ZERO ENERGY SCATTERING FOR ONE DIMENSIONAL SCHRÖDINGER OPERATORS WITH INVERSE SQUARE POTENTIALS

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ABSTRACT. This paper studies the scattering matrix $\mathbb{S}(E; \hbar)$ of the problem

$$-\hbar^2 \psi''(x) + V(x)\psi(x) = E\psi(x)$$

for positive potentials $V \in C^\infty(\mathbb{R})$ with inverse square behavior as $x \rightarrow \pm\infty$. It is shown that each entry takes the form $\mathbb{S}_{ij}(E; \hbar) = \mathbb{S}_{ij}^{(0)}(E; \hbar)(1 + \hbar\sigma_{ij}(E; \hbar))$ where $\mathbb{S}_{ij}^{(0)}(E; \hbar)$ is the WKB approximation relative to the *modified potential* $V(x) + \frac{\hbar^2}{4}\langle x \rangle^{-2}$ and the correction terms σ_{ij} satisfy $|\partial_E^k \sigma_{ij}(E; \hbar)| \leq C_k E^{-k}$ for all $k \geq 0$ and uniformly in $(E, \hbar) \in (0, E_0) \times (0, \hbar_0)$ where E_0, \hbar_0 are small constants. This asymptotic behavior is not universal: if $-\hbar^2 \partial_x^2 + V$ has a *zero energy resonance*, then $\mathbb{S}(E; \hbar)$ exhibits different asymptotic behavior as $E \rightarrow 0$. The resonant case is excluded here due to $V > 0$.

1. INTRODUCTION

This paper revisits the much studied problem of determining the reflection and transmission coefficients for semi-classical operators of the form

$$(1.1) \quad P(x, \hbar D) := -\hbar^2 \frac{d^2}{dx^2} + V(x)$$

where V is real-valued and assumed to decay at infinity. There are two atypical features of this work, at least relative to the existing literature on this topic:

- (i) we wish to understand the zero energy limit, in fact uniformly¹ in small \hbar
- (ii) the smooth potential V decays like an inverse square at both ends²

We remark that (i) and (ii) are closely related. Indeed, the $\langle x \rangle^{-2}$ decay is “critical” with respect to the zero energy limit in the sense that $\langle x \rangle^{-2-\epsilon}$ is easier and behaves very differently. In the semi-classical literature it is more customary to encounter the criticality of the Coulomb decay $\langle x \rangle^{-1}$; the reason for this is that the Coulomb decay is critical for *positive energies*. Note that the numerology around these decay rates applies to all dimensions and not just to one dimension. The motivation

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¹More precisely, the asymptotic analysis is carried out up to multiplicative errors of the form $1 + O(\hbar)$ where the $O(\hbar)$ needs to be uniform in small energies.

²The methods of this paper also apply to the case where the potential exhibits inverse square decay as $x \rightarrow \infty$ and some other decay as $x \rightarrow -\infty$; for that, one of course needs to be able to carry out the scattering theory on $x < 0$. If the decay is $|x|^{-\alpha}$ with $0 < \alpha < 2$, then [25] applies, whereas for $\alpha > 2$ one can use classical scattering methods.

for considering this particular problem comes from several sources. First, smooth potentials which behave as an inverse square at one or both ends arise in several contexts in physics and geometry, for example in general relativity in connection with Schwarzschild and de-Sitter spaces, see [6]. Second, this paper is part of the program initiated in [23] and [24]. In fact, the analysis carried out here is an essential part in the solution of the “large angular momentum” problem from [24].

Let us briefly review some elementary features of scattering, cf. [7] and [17]: For simplicity, let $\hbar = 1$ for now and write $H = P(x, D)$. Recall that the Jost solutions $f_{\pm}(x; \lambda)$ are required to satisfy

$$Hf_{\pm}(\cdot, \lambda) = \lambda^2 f_{\pm}(\cdot, \lambda), \quad f_{\pm}(x, \lambda) \sim e^{\pm i\lambda x} \quad \text{as } x \rightarrow \pm\infty$$

Provided $V \in L^1$ and $\lambda \neq 0$ they exist and are uniquely determined as solutions of the Volterra equation

$$(1.2) \quad f_+(x, \lambda) = e^{ix\lambda} + \int_x^\infty \frac{\sin(\lambda(y-x))}{\lambda} V(y) f_+(y, \lambda) dy$$

and similarly for $f_-(\cdot, \lambda)$. The resolvent kernel of H can now be expressed in the form

$$(H - (\lambda^2 + i0))^{-1}(x, y) = \frac{f_+(x, \lambda)f_-(y, \lambda)}{W(\lambda)} \chi_{[x>y]} + \frac{f_+(y, \lambda)f_-(x, \lambda)}{W(\lambda)} \chi_{[x<y]}$$

for all $\lambda > 0$ where $W(\lambda) = W(f_+(\cdot, \lambda), f_-(\cdot, \lambda))$. The reflection and transmission coefficients are defined by the relations

$$\begin{aligned} t_+(\lambda)f_+(\cdot, \lambda) &= r_+(\lambda)f_-(\cdot, \lambda) + f_-(\cdot, -\lambda) \\ t_-(\lambda)f_-(\cdot, \lambda) &= r_-(\lambda)f_+(\cdot, \lambda) + f_+(\cdot, -\lambda) \end{aligned}$$

and satisfy

$$(1.3) \quad t_- = t_+, \quad 1 = |t_+|^2 + |r_+|^2 = |t_-|^2 + |r_-|^2, \quad r_- = -\bar{r}_+ t / \bar{t}$$

For fixed $\lambda > 0$, consider the following bases of the space of solutions to the equation $Hf = \lambda^2 f$:

$$(f_+(\cdot, \lambda), f_-(\cdot, \lambda)), \quad (f_+(\cdot, -\lambda), f_-(\cdot, -\lambda))$$

The former is referred to as *outgoing* and the latter as *incoming*. In that case the matrix $\mathbb{S}(\lambda)$ which transforms the coefficients of a solution relative to these bases satisfies

$$\mathbb{S}(\lambda) = \begin{bmatrix} t(\lambda) & r_-(\lambda) \\ r_+(\lambda) & t(\lambda) \end{bmatrix}$$

It is called the *scattering matrix* and is unitary. Of special interest to us is the behavior as $\lambda \rightarrow 0+$. Note that if

$$\int_{-\infty}^\infty \langle x \rangle |V(x)| dx < \infty$$

then $f_+(x, \lambda) \rightarrow f_+(x, 0)$ as $\lambda \rightarrow 0$ where the latter satisfies the limit equation of (1.2), viz.

$$f_+(x, 0) = 1 + \int_x^\infty (y-x)V(y)f_+(y, 0) dy$$

It is known that $\mathbb{S}(\lambda)$ is continuous in $\lambda \geq 0$ under this moment condition, see [12]. To describe the possible values of $\mathbb{S}(0)$, recall that H has a *zero energy resonance* iff $f_{\pm}(\cdot, 0)$ are linearly dependent or, equivalently, iff $W(0) = 0$. Furthermore, since

$t(\lambda) = -\frac{2i\lambda}{W(\lambda)}$ this is equivalent to $t(0) \neq 0$. In conclusion, if zero energy is not resonant, then

$$\mathbb{S}(0) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

whereas in the resonant case

$$\mathbb{S}(0) = \begin{bmatrix} t & -r \\ r & t \end{bmatrix}$$

for some real $r, t \in [-1, 1]$, $t \neq 0$.

If $\langle x \rangle V(x) \notin L^1(\mathbb{R})$, then the behavior of $\mathbb{S}(\lambda)$ as $\lambda \rightarrow 0$ is completely different. In this paper, we focus on the border line case of positive inverse square potentials for (1.1) and \hbar small (for the remainder of the paper, we now let \hbar be a small positive quantity). It is precisely this case which arises in the geometric problem considered in [23], [24]. Our main theorem is as follows. We denote the energy by $E = \lambda^2 > 0$, see above, and the scattering matrix of (1.1) by

$$\mathbb{S}(E; \hbar) = \begin{bmatrix} t(E; \hbar) & r_-(E; \hbar) \\ r_+(E; \hbar) & t(E; \hbar) \end{bmatrix} = \begin{bmatrix} \mathbb{S}_{11}(E; \hbar) & \mathbb{S}_{12}(E; \hbar) \\ \mathbb{S}_{21}(E; \hbar) & \mathbb{S}_{22}(E; \hbar) \end{bmatrix}$$

In view of (1.3) it suffices to describe the first row of this matrix. In this paper, $O(\cdot)$ terms will be differentiable functions and we will typically state bounds on their derivatives with regard to the relevant variables depending on the context.

Theorem 1. *Let $V \in C^\infty(\mathbb{R})$ with $V > 0$ and $V(x) = \mu_\pm^2 x^{-2} + O(x^{-3})$ as $x \rightarrow \pm\infty$ where $\mu_+ \neq 0$, $\mu_- \neq 0$ and $\partial_x^k O(x^{-3}) = O(x^{-3-k})$ for all $k \geq 0$. Denote*

$$(1.4) \quad V_0(x; \hbar) := V(x) + \frac{\hbar^2}{4} \langle x \rangle^{-2}$$

and let $E_0 > 0$ be such that for all $0 < E < E_0$ and $0 < \hbar < 1$, $V_0(x; \hbar) = E$ has a unique pair of solutions, which we denote by $x_2(E; \hbar) < 0 < x_1(E; \hbar)$. Define

$$(1.5) \quad \begin{aligned} S(E; \hbar) &:= \int_{x_2(E; \hbar)}^{x_1(E; \hbar)} \sqrt{V_0(y; \hbar) - E} dy \\ T_+(E; \hbar) &:= x_1(E; \hbar) \sqrt{E} - \int_{x_1(E; \hbar)}^{\infty} (\sqrt{E - V_0(y; \hbar)} - \sqrt{E}) dy \\ T_-(E; \hbar) &:= -x_2(E; \hbar) \sqrt{E} - \int_{-\infty}^{x_2(E; \hbar)} (\sqrt{E - V_0(y; \hbar)} - \sqrt{E}) dy \end{aligned}$$

as well as $T(E; \hbar) := T_+(E; \hbar) + T_-(E; \hbar)$. Then for all $0 < \hbar < \hbar_0$ where $\hbar_0 = \hbar_0(V) > 0$ is small and $0 < E < E_0$

$$(1.6) \quad \begin{aligned} \mathbb{S}_{11}(E; \hbar) &= e^{-\frac{1}{\hbar}(S(E; \hbar) + iT(E; \hbar))} (1 + \hbar \sigma_{11}(E; \hbar)) \\ \mathbb{S}_{12}(E; \hbar) &= -ie^{-\frac{2i}{\hbar}T_+(E; \hbar)} (1 + \hbar \sigma_{12}(E; \hbar)) \end{aligned}$$

where the correction terms satisfy the bounds

$$(1.7) \quad |\partial_E^k \sigma_{11}(E; \hbar)| + |\partial_E^k \sigma_{12}(E; \hbar)| \leq C_k E^{-k} \quad \forall k \geq 0,$$

with a constant C_k that only depends on k and V . The same conclusion holds if instead of (1.4) we were to define V_0 as $V_0 := V + \hbar^2 V_1$ with $V_1 \in C^\infty(\mathbb{R})$, $V_1(x; \hbar) = \frac{1}{4} \langle x \rangle^{-2} + O(x^{-3})$ as $x \rightarrow \pm\infty$ with $\partial_x^k O(x^{-3}) = O(x^{-3-k})$ for all $k \geq 0$ and uniformly in $0 < \hbar \ll 1$.

The addition of $\frac{\hbar^2}{4}\langle x \rangle^{-2}$ to $V(x)$ is crucial and similar to the “Langer modification”, see for example [9]; indeed, if we were to use V instead of V_0 in (1.5), then the bounds (1.7) would fail due to a factor of $\log E$ as $E \rightarrow 0$. This is in contrast to potentials decaying like $|x|^{-\alpha}$ with $0 < \alpha < 2$ for which the modification is not needed, i.e., the usual WKB ansatz works, see [25]. On the other hand, note that as long as $E_0 > E > \varepsilon > 0$ the turning points $x_j(E; \hbar)$ will remain bounded and the distinction between V_0 and V is therefore moot. Indeed, the effect of passing from V to V_0 and vice versa is merely a harmless factor of the form $1 + O(\hbar)$ where the $O(\cdot)$ term of course depends on ε . In the range $E_0 > E > \varepsilon > 0$ Theorem 1 is well-known and classical. See for example Chapter 13 of [19] as well as Ramond’s work [22] for a more recent reference (Ramond, however, is more concerned with the scattering problem for energies close to the maximum of a barrier and he also assumes that the potential is dilation analytic).

We remark that the infinite differentiability assumption on V can be relaxed to some finite amount of smoothness (in which case we can only ask for correspondingly many derivatives with respect to E), but we do not elaborate on this issue here. A more substantial problem is that of relaxing the positivity assumption. We conjecture that $V > 0$ can be replaced by the strictly weaker assumption that *zero energy is not a resonance* of $P(x, \hbar D)$. Recall the definition of a zero energy resonance in this context, cf. [3], [25], and Section 3 of [24]: it means that the two subordinate zero-energy solutions at $\pm\infty$ are linearly dependent (a “subordinate solution” at either end refers to the nonzero solution of $P(x, \hbar D)f = 0$ with the slowest possible growth at that end; it is unique up to a nonzero scalar factor).

Note, however, that some condition is needed in Theorem 1; indeed, in [24] it was shown that for operators of the form considered in Theorem 1 with $\mu_+^2 = \mu_-^2 = \nu^2 - \frac{1}{4}$, $\nu > \frac{1}{2}$, and $\hbar = 1$

$$(1.8) \quad W(E; \hbar) \sim E^{\frac{1}{2}-\nu}(W_0 + O(E^\varepsilon)) \quad \text{as } E \rightarrow 0+$$

for some $W_0 \neq 0$ and $\varepsilon > 0$ *provided there is no zero energy resonance*. In the resonant case, it was shown in [24] that $W_0 = 0$. The following relation between \mathbb{S}_{11} in (1.6) and the Wronskian $W(E; \hbar)$

$$(1.9) \quad W(E; \hbar) = \frac{-2i\sqrt{E}}{\hbar\mathbb{S}_{11}(E; \hbar)}$$

allows one to deduce (1.8) with $W_0 \neq 0$ from Theorem 1 (note that for inverse square potentials $S(E; \hbar)$ behaves like $|\log E|$ so that the apparent exponential behavior in (1.6) turns into a power-law in E). This deduction also proves that Theorem 1 necessarily fails in the presence of a zero energy resonance. Another aspect of (1.8) concerns the case of *large* \hbar , say $\hbar = 1$. Indeed, it shows that Theorem 1 gives the correct behavior of the scattering matrix even in that case, but then the energy takes over as the small parameter.

This paper is organized as follows. Section 2 constructs a fundamental system of zero energy solutions to (1.1) via the usual WKB ansatz but for V_0 rather than for V . Since we require uniform bounds in Theorem 1 as $E \rightarrow 0$, the construction of Jost solutions for positive energies which is carried out in Section 3 needs to yield the zero energy solutions in the limit $E \rightarrow 0$. We choose to reverse this process and show that V_0 is precisely the right potential to use in the WKB method at zero

energy. The logic is simple: the WKB ansatz

$$\psi_{0,\pm}(x) := V^{-\frac{1}{4}}(x) \exp\left(\pm \hbar^{-1} \int_{x_0}^x \sqrt{V(y)} dy\right)$$

satisfies an equation of the form

$$(-\hbar^2 \partial_x^2 + V)\psi_{0,\pm} = \hbar^2(-x^{-2}/4 + O(x^{-3}))\psi_{0,\pm}$$

where the $x^{-2}/4$ term on the right-hand side is universal for all potentials that have an inverse square decay as $x \rightarrow \infty$ as specified in Theorem 1. Since this term has the same decay as V we need to bring it to the left-hand side leading to our choice of V_0 .

The main technical work of this paper is carried out in Section 3. It is here that the (semi-classical) Jost solutions are constructed for all energies in the range $0 < E < E_0$. We use Langer's method which is based on the Liouville-Green transform, see Chapters 6 and 11 in [19]: switching to the new independent variable

$$\zeta = \zeta(x, E; \hbar) := \text{sign}(x - x_1(E; \hbar)) \left| \frac{3}{2} \int_{x_1(E; \hbar)}^x \sqrt{|V_0(x; \hbar) - E|} d\eta \right|^{\frac{2}{3}}, \quad x \geq 0$$

and to the new dependent variable $w(\zeta) = \sqrt{\zeta} f$ reduces $P(x, \hbar D)f = Ef$, see (1.1), to an Airy equation perturbed by a potential of size \hbar^2 . It is here that $V > 0$ becomes relevant: it ensures that for all small $E > 0$ there is a unique turning point $x_1(E) > 0$ and that $V_0(x; \hbar) > E$ for all $0 < x < x_1(E; \hbar)$. Hence we can cover $x \geq 0$ by the intervals $\zeta(0, E; \hbar) < \zeta \leq 0$ and $\zeta \geq 0$. In each of these intervals we solve the perturbed Airy equations up to multiplicative errors of the form $1 + O(\hbar)$ where the $O(\cdot)$ term is uniform in E . It is in the range $\zeta(0, E; \hbar) < \zeta \leq 0$ that the choice of V_0 (rather than V) becomes decisive; this of course is to be expected as this range turns into the whole interval $x \geq 0$ as $E \rightarrow 0$ and WKB applied to V instead of V_0 fails at $E = 0$, see Section 2. Theorem 1 is proved in Section 5 by evaluating the Wronskians

$$W(f_+(\cdot, E), f_-(\cdot, E)), \quad W(f_+(\cdot, E), \overline{f_-(\cdot, E)})$$

at $x = 0$. Section 6 discusses the range of validity of Theorem 1 as the energy increases towards a unique non-degenerate maximum of a barrier potential. Finally, the appendix describes a certain “normal-form” reduction of (1.1) to a Bessel equation on a region containing the turning point. Even though we do not base our asymptotic analysis on this reduction (but rather the Airy equation), we still believe that this is of independent interest.

Needless to say, there is a vast literature related to the semi-classical analysis of the Schrödinger equation and it is impossible to do any justice to it here. Somewhat curiously, however, there does not seem to be any literature on potentials which are globally smooth on the line and which exhibit inverse square decay. On the other hand, potentials which are *exactly* inverse square are of course ubiquitous, especially in the physics literature. For a recent paper in this direction involving WKB see [9] and for a time-dependent analysis see the recent papers [20], [21], as well as [4], [5] and the references cited there. Potentials which decay of the form $|x|^{-\alpha}$, $0 < \alpha < 2$, have been studied with similar objectives as here, see [13], [18] and [25]. For other work on low energies see [2], [3], [8], and [26], as well as [10].

2. ZERO ENERGY SOLUTIONS

In order to motivate the choice of V_0 in Theorem 1 we will now obtain a fundamental system for the equation

$$(2.1) \quad -\hbar^2 f''(x) + V(x)f(x) = 0$$

on the half axis $x > x_0$. Here we assume that $V(x) = \mu^2 x^{-2} + O(x^{-3})$ with $\mu > 0$ as $x \rightarrow \infty$ and x_0 is chosen so large that $V(x) > 0$ for $x > x_0$. As before, we require $\partial_x^k O(x^{-3}) = O(x^{-3-k})$ for all $k \geq 0$. Ignoring the $O(\cdot)$ term, we have on the one hand

$$(-\hbar^2 \partial_x^2 + \mu^2 x^{-2})x^{\frac{1}{2} \pm \alpha} = 0, \quad \alpha^2 = \frac{1}{4} + \mu^2 \hbar^{-2}$$

On the other hand, with $Q(x; \hbar) := \mu^2 x^{-2} + \hbar^2 x^{-2}/4$,

$$Q^{-\frac{1}{4}}(x; \hbar) e^{\pm \frac{1}{\hbar} \int_{x_0}^x \sqrt{Q(y; \hbar)} dy} = c x^{\frac{1}{2} \pm \alpha}$$

with some $c \neq 0$. This motivates the following result.

Proposition 2. *On $x > x_0$ a fundamental system of solutions for (2.1) is given by*

$$(2.2) \quad \psi_j(x; \hbar) = \tilde{\psi}_j(x; \hbar)(1 + \hbar a_j(x; \hbar)), \quad j = 1, 2$$

with

$$(2.3) \quad \begin{aligned} \tilde{\psi}_1(x; \hbar) &= V_0(x; \hbar)^{-\frac{1}{4}} e^{\frac{1}{\hbar} S(x; \hbar)} \\ \tilde{\psi}_2(x; \hbar) &= V_0(x; \hbar)^{-\frac{1}{4}} e^{-\frac{1}{\hbar} S(x; \hbar)}, \end{aligned}$$

where $V_0(x; \hbar) = V(x) + \frac{\hbar^2}{4} \langle x \rangle^{-2}$, $S(x; \hbar) = \int_{x_0}^x \sqrt{V_0(t; \hbar)} dt$ and

$$(2.4) \quad \sup_{0 < \hbar < 1} |\partial_x^\ell a_j(x; \hbar)| \leq C_{\ell, \mu} x^{-\ell}$$

for $x > x_0$, $j = 1, 2$ and $\ell = 0, 1$. Their Wronskian satisfies

$$(2.5) \quad W(\psi_1, \psi_2) = \frac{-2}{\hbar} (1 + O(\hbar)).$$

as $\hbar \rightarrow 0$.

Proof. Let us consider the case of ψ_1 . Hence, we need to find a_1 so that ψ_1 is a solution to the differential equation

$$(2.6) \quad -\hbar^2 u''(x) + V(x)u(x) = 0.$$

Substituting the first expression of (2.2) into the differential equation (2.6) yields

$$(2.7) \quad -\hbar^2 [\tilde{\psi}_1''(1 + \hbar a_1) + 2\hbar \tilde{\psi}_1' a_1' + \hbar \tilde{\psi}_1 a_1''] + V \tilde{\psi}_1 (1 + \hbar a_1) = 0$$

Setting $V_2 := \frac{1}{4} \langle x \rangle^{-2} - \frac{1}{4} \frac{V_0''}{V_0} + \frac{5}{16} \frac{V_0'^2}{V_0^2}$ and observing that $-\hbar^2 \tilde{\psi}_1'' + V \tilde{\psi}_1 = -\hbar^2 V_2 \tilde{\psi}_1$, we deduce after dividing the equation by $\tilde{\psi}_1$

$$(2.8) \quad -(1 + \hbar a_1) V_2 = \hbar (a_1'' + 2 \frac{\tilde{\psi}_1'}{\tilde{\psi}_1} a_1'),$$

We now note the following essential feature of V_2 (which was the reason for defining V_0 as above):

$$|V_2(x)| \leq C x^{-3}, \quad |\partial_x^k V_2(x)| \leq C_k x^{-3-k} \quad \forall k \geq 0$$

To solve (2.8) we multiply both sides by $\tilde{\psi}_1^2$ and obtain

$$(2.9) \quad (a_1' \tilde{\psi}_1^2)' = \frac{-1}{\hbar} V_2 \tilde{\psi}_1^2 - a_1 V_2 \tilde{\psi}_1^2.$$

Integration and using the definition of the $\tilde{\psi}_1$ yield

$$(2.10) \quad \begin{aligned} a_1'(x) &= \frac{1}{\hbar} \int_x^{x_0} V_2(y) \tilde{\psi}_1^{-2}(x) \tilde{\psi}_1^2(y) dy + \int_x^{x_0} a_1(y) V_2(y) \tilde{\psi}_1^{-2}(x) \tilde{\psi}_1^2(y) dy \\ &= \frac{1}{\hbar} \int_x^{x_0} V_0(x)^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x))} V_2(y) dy \\ &\quad + \int_x^{x_0} V_0(x)^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x))} V_2(y) a_1(y) dy. \end{aligned}$$

Strictly speaking, $a_1 = a_1(x, \hbar)$ but we suppress the \hbar from the notation here. After integration in (2.10) we obtain

$$(2.11) \quad \begin{aligned} a_1(x) &= \frac{-1}{\hbar} \int_x^{x_0} \int_{x'}^{x_0} V_0(x')^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x'))} V_2(y) dy dx' \\ &\quad - \int_x^{x_0} \int_{x'}^{x_0} V_0(x')^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x'))} V_2(y) a_1(y) dy dx' \\ &= \frac{-1}{\hbar} \int_x^{x_0} \int_x^y V_0(x')^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x'))} V_2(y) dx' dy \\ &\quad - \int_x^{x_0} \int_x^y V_0(x')^{\frac{1}{2}} V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}(S(y)-S(x'))} V_2(y) a_1(y) dx' dy. \end{aligned}$$

Furthermore,

$$(2.12) \quad \int_x^y V_0(x')^{\frac{1}{2}} e^{\frac{-2}{\hbar}S(x')} dx' = -\frac{\hbar}{2} [e^{\frac{-2}{\hbar}S(y)} - e^{\frac{-2}{\hbar}S(x)}].$$

and

$$(2.13) \quad V_0(y)^{\frac{-1}{2}} e^{\frac{2}{\hbar}S(y)} \int_x^y V_0(x')^{\frac{1}{2}} e^{\frac{-2}{\hbar}S(x')} dx' = \frac{-\hbar}{2} V_0(y)^{\frac{-1}{2}} [1 - e^{\frac{2}{\hbar}(S(y)-S(x))}].$$

From this it follows that

$$(2.14) \quad a_1(x) = \frac{1}{2} \int_{x_0}^x V_0(y)^{\frac{-1}{2}} [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] V_2(y) [1 + \hbar a_1(y)] dy.$$

This is a standard Volterra equation. To solve it, we first introduce a new function $\rho(x)$ given by

$$(2.15) \quad \rho(x) = \int_{x_0}^x |V_0(y)^{\frac{-1}{2}} V_2(y)| dy.$$

In view of the decay of V_2 we see that the integrand here decays like y^{-2} so that $\rho \in L^\infty(x_0, \infty)$. Then we define a sequence $a_1^s(x)$, $s = 0, 1, \dots$, with $a_1^0(x) = 0$ and

$$(2.16) \quad a_1^s(x) = \frac{1}{2} \int_{x_0}^x V_0(y)^{\frac{-1}{2}} [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] V_2(y) [1 + \hbar a_1^{s-1}(y)] dy$$

We claim that

$$(2.17) \quad |a_1^s(x) - a_1^{s-1}(x)| \leq \frac{\rho^s(x) \hbar^{s-1}}{s!}.$$

To prove this we proceed by induction and observe that

$$S(y) - S(x) = - \int_y^x V_0^{\frac{1}{2}}(t, \hbar) dt$$

and hence for $0 < x_0 < y < x$, $|e^{\frac{2}{\hbar}(S(y)-S(x))} - 1| \leq 2$. Therefore, (2.17) is valid for $s = 1$. Furthermore, if we assume the validity of (2.17) for $s = k$ then since (2.18)

$$a_1^{k+1}(x) - a_1^k(x) = \frac{\hbar}{2} \int_x^{x_0} V_0(y)^{\frac{-1}{2}} [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] V_2(y) (a_1^k(y) - a_1^{k-1}(y)) dy,$$

we have

$$(2.19) \quad \begin{aligned} |a_1^{k+1}(x) - a_1^k(x)| &\leq \frac{\hbar^k}{k!} \int_x^{x_0} |V_0(y)^{\frac{-1}{2}} V_2(y)| \rho^k(y) dy \\ &= -\frac{\hbar^k}{k!} \int_x^{x_0} \rho'(y) \rho^k(y) dy = \frac{\rho^{k+1}(x) \hbar^k}{(k+1)!}. \end{aligned}$$

We would like to have an estimate for the function $\rho(x)$, therefore it suffices that we obtain an estimate for $\int_{x_0}^x |V_0(y)|^{\frac{-1}{2}} |V_2(y)| dy$. As already noted above, for $x > x_0$

$$\rho(x) \leq \int_{x_0}^{\infty} |V_0(y)|^{\frac{-1}{2}} |V_2(y)| dy \leq C(\mu) < \infty$$

Hence, the solution to the integral equation (2.14) is given by

$$(2.20) \quad a_1(x) = \sum_{s=1}^{\infty} (a_1^s(x) - a_1^{s-1}(x))$$

and satisfies $\sup_{x > x_0} |a_1(x)| \leq C(\mu) < \infty$ uniformly in $0 < \hbar < 1$. To derive the estimate for $a_1'(x)$, we observe that

$$\frac{1}{\hbar} e^{\frac{2}{\hbar}(S(y)-S(x))} = \frac{1}{2} V_0^{\frac{-1}{2}}(y) \partial_y e^{\frac{2}{\hbar}(S(y)-S(x))}.$$

Therefore, using this observation and integrating by parts in (2.10) yields

$$\begin{aligned} a_1'(x) &= \frac{1}{2} \int_{x_0}^x \left(\frac{V_0(x)}{V_0(y)} \right)^{\frac{1}{2}} \partial_y [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] V_0(y)^{\frac{-1}{2}} V_2(y) dy \\ &\quad + \int_{x_0}^x [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] \left(\frac{V_0(x)}{V_0(y)} \right)^{\frac{1}{2}} V_2(y) a_1(y) dy \\ &= -\frac{1}{2} \int_{x_0}^x V_0(x)^{\frac{1}{2}} [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] \partial_y [V_0(y)^{-1} V_2(y)] dy \\ &\quad - \frac{1}{2} [e^{\frac{2}{\hbar}(S(x_0)-S(x))} - 1] \left(\frac{V_0(x)}{V_0(x_0)} \right)^{\frac{1}{2}} V_2(x_0) \\ &\quad + \int_{x_0}^x [e^{\frac{2}{\hbar}(S(y)-S(x))} - 1] \left(\frac{V_0(x)}{V_0(y)} \right)^{\frac{1}{2}} V_2(y) a_1(y) dy. \end{aligned}$$

At this point we note that for $x > x_0$ (x_0 large enough), we have $|\partial_x^\ell V_0(x, \hbar)| \leq c_{\ell, \mu} x^{-2-\ell}$, uniformly in \hbar . Hence

$$(2.21) \quad |\partial_y [V_0(y)^{-1} V_2(y)]| \lesssim y^{-2},$$

and using the boundedness of a_1 , together with that of $\rho(x)$, obviously implies that for $x > x_0$

$$|a_1'(x)| \leq C_\mu x^{-1},$$

uniformly in $0 < \hbar < 1$ as desired. For the case of a_2 one proceeds in essentially the same way using, however, the forward Green function rather than the backward one. This yields

$$a_2(x) = \frac{1}{2} \int_x^\infty V_0(y)^{\frac{-1}{2}} [e^{\frac{-2}{\hbar}(S(y)-S(x))} - 1] V_2(y) [1 + \hbar a_2(y)] dy.$$

The same arguments as before now show that a_2 satisfies (2.4). The Wronskian $W(\psi_1, \psi_2)$ is obtain by evaluating at $x = \infty$. \square

The same analysis of course yields zero energy solutions with the correct asymptotic behavior as $x \rightarrow -\infty$. Note that the solution ψ_2 is the sub-ordinate one, i.e., it is the unique (up to a nonzero scalar multiple) solution with the slowest possible growth. Hence, a zero energy resonance in the context of Theorem 1 would mean the existence of a nonzero solution to $P(x, \hbar D)f = 0$ with $f(x) \sim c_\pm |x|^{\frac{1}{2}-\alpha_\pm}$ as $x \rightarrow \pm\infty$ and with

$$\alpha_\pm = \sqrt{\frac{1}{4} + \mu_\pm^2 \hbar^{-2}}$$

It is easy to see that such a solution cannot exist if $V > 0$. Indeed, let χ be a standard cut-off function with $\chi(0) = 1$ and set $\chi_R(x) := \chi(x/R)$. If a globally subordinate solution $f(x)$ did exist, then

$$0 = \limsup_{R \rightarrow \infty} \langle P(\cdot, \hbar D)f, \chi_R f \rangle = \int [(f')^2(x) + V(x)f^2(x)] dx$$

implies that $f = 0$, which is a contradiction.

3. THE LIOUVILLE-GREEN TRANSFORM FOR SMALL ENERGIES

In this section, we consider the equation

$$(3.1) \quad -\hbar^2 f_\pm''(x) + V(x)f_\pm(x) = Ef_\pm(x)$$

where V is as in Theorem 1. As explained in the introduction, we will use the Liouville-Green transform to reduce (3.1) to a perturbed Airy equation. We begin with a statement of the formal aspects (i.e., not involving estimates) of this transform, cf. Chapter 6 in [19] and Langer's papers [14]–[16]. Henceforth, V, V_0 are as in Theorem 1. Throughout this section $x \geq 0$.

Lemma 3. *There exists $E_0 = E_0(V) > 0$ so that for all $0 < E < E_0$ one has the following properties: the equation $V_0(x; \hbar) - E = 0$ has a unique (simple) solution on $x > 0$ which we denote by $x_1 = x_1(E; \hbar)$. With $Q_0 := V_0 - E$*

$$(3.2) \quad \zeta = \zeta(x, E; \hbar) := \text{sign}(x - x_1(E; \hbar)) \left| \frac{3}{2} \int_{x_1(E; \hbar)}^x \sqrt{|Q_0(u, E; \hbar)|} du \right|^{\frac{2}{3}}$$

defines a smooth change of variables $x \mapsto \zeta$ for all $x \geq 0$. Let $q := -\frac{Q_0}{\zeta}$. Then $q > 0$, $\frac{d\zeta}{dx} = \zeta' = \sqrt{q}$, and

$$-\hbar^2 f'' + (V - E)f = 0$$

transforms into

$$(3.3) \quad -\hbar^2 \ddot{w}(\zeta) = (\zeta + \hbar^2 \tilde{V}(\zeta, E; \hbar))w(\zeta)$$

under $w = \sqrt{\zeta'} f = q^{\frac{1}{4}} f$. Here $\dot{\cdot} = \frac{d}{d\zeta}$ and

$$\tilde{V} := \frac{1}{4} q^{-1} \langle x \rangle^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2}$$

Proof. Let $E_0 > 0$ be such that $V_0(x; \hbar) = E$ has a unique pair of solutions denoted by $x_2(E; \hbar) < 0 < x_1(E; \hbar)$. It is clear that (3.2) defines a smooth map away from $x = x_1(E; \hbar)$. Taylor-expanding $Q_0(x, E; \hbar)$ in a neighborhood of that point and using that $V_0'(x_1(E; \hbar)) < 0$ implies that $\zeta(x, E; \hbar)$ is smooth around $x = x_1$ as well with $\zeta'(x_1, E; \hbar) > 0$. Next, one checks that

$$\dot{w} = q^{-\frac{1}{4}} f' + \frac{dq^{\frac{1}{4}}}{d\zeta} f, \quad \ddot{w} = q^{-\frac{3}{4}} f'' + \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} f$$

and thus, using $-\hbar^2 f'' = (E - V)f$,

$$\begin{aligned} -\hbar^2 \ddot{w} &= q^{-1} (E - V)w - \hbar^2 q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} w \\ &= q^{-1} (-Q_0 + \hbar^2 \langle x \rangle^{-2} / 4) w - \hbar^2 q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} w \\ &= \zeta w(\zeta) + \hbar^2 \left(q^{-1} \langle x \rangle^{-2} / 4 - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} \right) w \end{aligned}$$

as claimed. \square

We now analyze the properties of the change of variables introduced in the previous lemma. Recall that

$$V_0(x; \hbar) = (\mu_+^2 + \hbar^2 / 4) x^{-2} (1 + O(x^{-1}))$$

which implies that

$$(3.4) \quad x_1(E; \hbar) = c(\hbar) E^{-\frac{1}{2}} (1 + O(E^{\frac{1}{2}})), \quad c(\hbar) = \sqrt{\mu_+^2 + \hbar^2 / 4}$$

It will be convenient for us to normalize the constants here so that $c(\hbar) = \sqrt{2}$ and we shall assume that for the remainder of this section. Moreover, we shall mostly suppress the harmless \hbar dependence of various functions in our notation. We begin with the following “normal form” lemma which will allow us to describe the function ζ in any region of the form $x \geq \varepsilon E^{-\frac{1}{2}}$ (which, in particular, contains the turning point). Note that Lemma 4 normalizes the turning point x_1 to $\xi = 1$ by scaling out the energy.

Lemma 4. *Let $\varepsilon > 0$ but fixed. There exists a smooth map $\xi = \xi(y, E)$ on $(y, E) \in (\varepsilon, \infty) \times (0, E_0)$ with E_0 small so that $\xi(E^{\frac{1}{2}} x_1(E), E) = 1$ and for all (y, E) in this range,*

$$(3.5) \quad 1 - E^{-1} V_0(E^{-\frac{1}{2}} y) = \left(\frac{d\xi}{dy} \right)^2 (1 - \xi^{-2})$$

and such that, with some constant $\xi_0(E)$,

$$(3.6) \quad \xi(y, E) = y + \xi_0(E) + y^{-1} \rho_0(y, E)$$

$$(3.7) \quad |\partial_E^k \partial_y^\ell \xi(y, E)| \leq C_{k,\ell} E^{-k} y^{1-\ell}$$

for all $k, \ell \geq 0$. The functions ξ_0 and ρ_0 from (3.6) satisfy

$$(3.8) \quad |\partial_E^k \xi_0(E)| \leq C_k E^{-k}, \quad |\partial_E^k \partial_y^j \rho_0(y, E)| \leq C_{jk} y^{-j} E^{-k}$$

for all $k, j \geq 0$ and uniformly in $(y, E) \in (1, \infty) \times (0, E_0)$. For fixed $0 < E < E_0$ the map $y \mapsto \xi(y, E)$ is a global diffeomorphism whose inverse $y = y(\xi, E)$ satisfies the bounds

$$(3.9) \quad y(\xi, E) = \xi + y_0(E) + \xi^{-1} \tilde{\rho}_0(\xi, E)$$

$$(3.10) \quad |\partial_E^k \partial_\xi^\ell y(\xi, E)| \leq C_{k,\ell} E^{-k} \xi^{1-\ell}$$

for all $k, \ell \geq 0$ and with functions $y_0, \tilde{\rho}_0$ satisfying (3.8) but relative to ξ rather than y . All constants are allowed to depend on $\varepsilon > 0$.

Proof. Set $y_1 = y_1(E) := E^{\frac{1}{2}} x_1(E)$. Then $y_1 = \sqrt{2} + O(E^{\frac{1}{2}})$ as $E \rightarrow 0+$. Note that the $O(\cdot)$ term here satisfies

$$\partial_E^k O(E^{\frac{1}{2}}) = O(E^{\frac{1}{2}-k}) \quad \forall k \geq 0$$

due to the corresponding assumption on the error term of V . The same comment applies to every $O(\cdot)$ term appearing in this proof, both with respect to derivatives in E and spatial variables³. Define the change of variables $\xi = \xi(y, E)$ via

$$(3.11) \quad \int_{y_1}^y \sqrt{1 - E^{-1} V_0(E^{-\frac{1}{2}} u)} du = \int_1^\xi \sqrt{1 - t^{-2}} dt \quad y > y_1$$

$$(3.12) \quad \int_y^{y_1} \sqrt{E^{-1} V_0(E^{-\frac{1}{2}} u) - 1} du = \int_\xi^1 \sqrt{t^{-2} - 1} dt \quad \varepsilon < y < y_1$$

By monotonicity, these identities define a unique correspondence between ξ and y on these ranges which is, moreover, smooth and strictly increasing on $\varepsilon < y < y_1$ and $y_1 < y < \infty$. By inspection, they also satisfy (3.5). Since

$$1 - E^{-1} V_0(E^{-\frac{1}{2}} u) = 1 - \frac{2}{u^2} + O(E^{\frac{1}{2}} u^{-3})$$

it follows furthermore that the interval $\varepsilon \leq y < \infty$ is transformed into one of the form $0 < \xi_1(E) < \xi < \infty$ where

$$(3.13) \quad \xi_1(E) = \xi_1(0) + O(E^{\frac{1}{2}}) \quad \text{as } E \rightarrow 0+$$

and $\xi_1(0) > 0$ is a constant. We first show that the map $\xi = \xi(y, E)$ so defined, is smooth for all $(y, E) \in (\varepsilon, 2) \times (0, E_0)$ together with the desired estimates. To this end, write

$$1 - E^{-1} V_0(E^{-\frac{1}{2}} y) = (y - y_1) U(y, E)$$

$$U(y, E) := -E^{-\frac{3}{2}} \int_0^1 V_0'(E^{-\frac{1}{2}}(y_1 + t(y - y_1))) dt$$

Then for all $0 < E \leq E_0$ and all $k, \ell \geq 0$,

$$(3.14) \quad \max_{1 \leq y \leq 2} |\partial_E^k \partial_y^\ell U(y, E)| \leq C_{k,\ell} E^{-k}, \quad \min_{1 \leq y \leq 2} U(y, E) \geq c_0 > 0$$

For all $\varepsilon < y < 2$ we rewrite (3.11) and (3.12) in the form

$$(3.15) \quad (y - y_1) Y(y, E) = (\xi - 1) X(\xi)$$

³We will say that a $O(\cdot)$ term behaves like a symbol if its derivatives are governed by such power-laws.

where

$$Y(y, E) := \left(\int_0^1 \sqrt{(1-t)U(y_1 + t(y - y_1), E)} dt \right)^{\frac{2}{3}}$$

$$X(\xi) := \left(\int_0^1 \frac{\sqrt{s(2 + s(\xi - 1))}}{1 + s(\xi - 1)} ds \right)^{\frac{2}{3}}$$

By the preceding,

$$|\partial_E^k \partial_y^\ell Y(y, E)| \leq C_{k,\ell} E^{-k}, \quad |\partial_\xi^j X(\xi)| \leq C_j \quad \forall k, \ell, j \geq 0$$

uniformly on the interval $\varepsilon \leq y \leq 2$ and the corresponding interval in ξ . By the inverse function theorem, (3.15) defines a (unique) smooth map also locally around $\xi = 1$ and $y = y_1$; this agrees with the previous definition for $y \neq y_1$ and thus furnishes the desired smooth extension through the point $y = y_1$. Furthermore, from

$$y_1 = \sqrt{2} + O(E^{\frac{1}{2}}), \quad \partial_E^k y_1 = O(E^{\frac{1}{2}-k})$$

and (3.15), (3.14) we conclude that

$$\max_{1 \leq y \leq 2} |\partial_E^k \partial_y^\ell \xi(y, E)| \leq C_{k,\ell} E^{-k}$$

for all $k, \ell \geq 0$, $0 < E < E_0$. For large y , we write

$$\int_{y_1}^y \left\{ 1 - \frac{2}{u^2} \left(1 + O(E^{\frac{1}{2}} u^{-1}) \right) \right\}^{\frac{1}{2}} du = \int_1^\xi \sqrt{1 - v^{-2}} dv$$

The integral on the right-hand side satisfies

$$\int_1^\xi \sqrt{1 - v^{-2}} dv = \xi + \kappa + O(\xi^{-1})$$

with a constant κ , whereas the one on the left-hand side is equal to

$$y + y_0 + O(y^{-1}) + O(E^{\frac{1}{2}} y^{-2})$$

with a constant $y_0(E)$. It is easy to see that

$$\xi + \kappa + O(\xi^{-1}) = y + y_0 + O(y^{-1}) + O(E^{\frac{1}{2}} y^{-2})$$

implies (3.6) and we are done. The statements about the inverse follow easily. \square

We refer to Lemma 4 as a “normal form” since (3.1), on the interval $x > E^{-\frac{1}{2}}$, turns into a suitably normalized (perturbed) Bessel equation in the variable ξ , see the appendix. By means of the change of variables introduced in Lemma 4 it is now an easy matter to describe $\zeta(x, E)$ from Lemma 3 on the interval $x \geq E^{-\frac{1}{2}}$.

Corollary 5. *For all $0 < E < E_0$ the following holds: there exists a constant $c_0 > \sqrt{2}$ so that on the interval $\varepsilon < \sqrt{E}x < c_0$*

$$(3.16) \quad \zeta(x, E) = 2^{\frac{1}{3}}(\xi - 1)[1 + O(\xi - 1)]$$

with $O(\cdot)$ analytic and $\xi = \xi(\sqrt{E}x, E)$. For all $x \geq E^{-\frac{1}{2}}c_0$,

$$(3.17) \quad \frac{2}{3}\zeta^{\frac{2}{3}}(x, E) = \xi + \gamma + O(\xi^{-1})$$

where γ is some constant and $O(\cdot)$ is analytic. Neither of the $O(\cdot)$ terms here depend on E (other than through ξ).

Proof. We begin with ξ close to $\xi = 1$. The action $S(x, E)$ then satisfies, with ξ as in Lemma 4,

$$\begin{aligned} S(x, E) &= \text{sign}(x - x_1(E)) \int_{x_1(E)}^x \sqrt{|E - Q_0(u)|} du \\ &= \text{sign}(x - x_1(E)) \int_{\sqrt{E}x_1(E)}^{\sqrt{E}x} \sqrt{|1 - E^{-1}V_0(E^{-\frac{1}{2}}y)|} dy \\ &= \text{sign}(\xi(\sqrt{E}x) - 1) \int_1^{\xi(\sqrt{E}x, E)} \sqrt{|1 - \eta^{-2}|} d\eta \\ &= \sqrt{2} \frac{2}{3} \text{sign}(\xi - 1) |\xi - 1|^{\frac{3}{2}} (1 + O(\xi - 1)) \end{aligned}$$

where the $O(\cdot)$ term is analytic in $|\xi - 1| < 1$ and $\xi = \xi(\sqrt{E}x, E)$. In terms of ζ this means that

$$\zeta(x, E) = 2^{\frac{1}{3}}(\xi - 1)[1 + O(\xi - 1)]$$

which is (3.16). The constant c_0 is chosen so that $1 < \xi(c_0 E^{-\frac{1}{2}}, E) < 2$ for all $0 < E < E_0$. Since $x_1(E) = \sqrt{2/E} + o(1)$ as $E \rightarrow 0$ we see that $c_0 > \sqrt{2}$. As for (3.17),

$$\begin{aligned} S(x, E) &= \int_1^{\xi(\sqrt{E}x, E)} \sqrt{1 - \eta^{-2}} d\eta \\ &= \int_1^{\xi} (1 + O(\eta^{-2})) d\eta = \xi + \gamma + O(\xi^{-1}) \end{aligned}$$

Since $\zeta = (\frac{3}{2}S)^{\frac{2}{3}}$, we are done. \square

In the region $0 < x < \varepsilon x_1(E)$ we have the following description of $\zeta(x, E)$ with ε the same as in Lemma 4. In fact, in the following lemma we will need ε small and then use this choice in Lemma 4.

Lemma 6. *For sufficiently small and fixed $\varepsilon > 0$ there exists a smooth function $\tilde{x}(x, E)$ on $0 \leq x \leq \varepsilon x_1(E)$ with*

$$\tilde{x}(x, E) = x(1 + O(Ex^2))$$

and such that

$$(3.18) \quad \frac{2}{3}\zeta^{\frac{3}{2}}(x, E) = \int_{\tilde{x}(x, E)}^{x_1(E)} \sqrt{V_0(v)} dv + O(E \log E)$$

for all $0 \leq x \leq \varepsilon x_1(E)$ and $0 < E < E_0$. The $O(\cdot)$ here behave like symbols.

Proof. Define \tilde{x} via

$$\begin{aligned} \int_0^{\tilde{x}(x, E)} \sqrt{V_0(v)} dv &= \int_0^x \sqrt{V_0(u) - E} du = \int_0^x \sqrt{V_0(u)} (1 + O(Eu^2)) du \\ &= \int_0^x \sqrt{V_0(u)} du + O(Ex^2) \end{aligned}$$

Provided $\varepsilon > 0$ is sufficiently small (independently of E , of course), it follows from monotonicity considerations that $\tilde{x}(x, E)$ exists with the desired properties. Next,

note that for all $0 < E < E_0$,

$$\int_0^{x_1(E)} \sqrt{V_0(u)} du - \int_0^{x_1(E)} \sqrt{V_0(u) - E} du = O(E \log E)$$

and thus

$$\begin{aligned} \frac{2}{3} \zeta^{\frac{3}{2}} &= \int_0^{x_1(E)} \sqrt{V_0(u) - E} du - \int_0^x \sqrt{V_0(u) - E} du \\ &= \int_0^{x_1(E)} \sqrt{V_0(u)} du - \int_0^{\tilde{x}(x,E)} \sqrt{V_0(v)} dv + O(E \log E) \\ &= \int_{\tilde{x}(x,E)}^{x_1(E)} \sqrt{V_0(v)} dv + O(E \log E) \end{aligned}$$

as claimed. \square

The point of (3.18) is that $O(E \log E)$ is negligible as compared to the integral on the right-hand side which is on the order of $|\log(E \langle x \rangle^2)| \gtrsim 1$. Thus, ζ behaves to leading order like $|\log(E \langle x \rangle^2)|^{\frac{2}{3}}$. We now turn to estimating the functions q, \tilde{V} from Lemma 3. In what follows, the notation $A \sim B$ will denote proportionality of $A, B > 0$ by some constants that are only allowed to depend on V . Also, $A \lesssim B$ will denote $A \leq CB$ where C is a constant, and similarly for $A \gtrsim B$.

Lemma 7. *Using the notations of Lemma 3, let $0 < E < E_0$. Then on the interval $\zeta \geq -1$ the functions $q = q(\zeta, E)$ and $\tilde{V} = \tilde{V}(\zeta, E)$ satisfy*

$$(3.19) \quad \begin{aligned} |\partial_E^k \partial_\zeta^\ell q| &\leq C_{k,\ell} E^{1-k} \langle \zeta \rangle^{-1-\ell} \\ |\partial_E^k \partial_\zeta^\ell \tilde{V}(\zeta, E)| &\leq C_{k,\ell} E^{-k} \langle \zeta \rangle^{-2-\ell} \quad \forall k, \ell \geq 0 \end{aligned}$$

On the interval $\zeta(0, E) \leq \zeta \leq -1$ we view q, \tilde{V} as functions of x via (3.2). Then one has $q \sim |\zeta|^{-1} \langle x \rangle^{-2}$ and there is the representation

$$(3.20) \quad \tilde{V}(\zeta, E) = -\frac{5}{16\zeta^2} + q^{-1}(E \beta_0(x, E) + \langle x \rangle^{-3} \beta_1(x, E))$$

where β_j satisfy the bounds

$$(3.21) \quad \begin{aligned} |\partial_E^k \partial_x^\ell \beta_j(x, E)| &\leq C_{k,\ell} E^{-k} \langle x \rangle^{-\ell} \quad j = 0, 1 \\ |\partial_E^k \partial_x^\ell q| &\leq C_{k,\ell} E^{-k} |\zeta|^{-1} \langle x \rangle^{-2-\ell} \quad \forall k, \ell \geq 0 \end{aligned}$$

All constants are independent of E .

Proof. The case $\zeta \geq -1$ corresponds to $x \geq \varepsilon x_1(E)$ by Lemma 4 and Corollary 5. We now use that corollary to write

$$\zeta = \zeta(\xi, E), \quad \xi = \xi(y, E), \quad y = E^{\frac{1}{2}} x$$

Then

$$q = (\zeta')^2 = E(\partial_\xi \zeta(\xi, E))^2 (\partial_y \xi(y, E))^2 \sim E$$

The derivative bounds on q now follow from those obtained in Lemma 4 and Corollary 5. As for \tilde{V} , we compute, with $\cdot = \frac{d}{d\zeta}$,

$$(3.22) \quad \tilde{V} = \frac{1}{4} q^{-1} \langle x \rangle^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d^2 \zeta} = \frac{1}{4} q^{-1} \langle x \rangle^{-2} + \frac{3}{16} q^{-2} \dot{q}^2 - \frac{1}{4} q^{-1} \ddot{q}$$

From the bounds on q which we just derived, the last two terms on the right-hand side of (3.22) are $\lesssim \zeta^{-2}$ and behave as stated under differentiation. To treat the first term, we invoke (3.9), (3.16), and (3.17) to write

$$\begin{aligned} q^{-1}\langle x \rangle^{-2} &= q^{-1}\langle E^{-\frac{1}{2}}y \rangle^{-2} = q^{-1}\langle E^{-\frac{1}{2}}y(\xi, E) \rangle^{-2} \\ &= q^{-1}\langle E^{-\frac{1}{2}}y(\xi(\zeta, E), E) \rangle^{-2} \end{aligned}$$

which implies the correct bounds. Indeed, if $|\zeta| \lesssim 1$, then $q \sim E$ and the change of variables $\zeta \mapsto \xi \mapsto y$ has derivatives of size $\lesssim 1$ relative to ζ uniformly in E . This implies that $q^{-1}\langle x \rangle^{-2} \sim 1$ with derivatives with respect to ζ of size $\lesssim 1$; furthermore, each derivative in E costs one power of E . Next, if $\zeta \geq 1$, then the change of variables $\zeta \mapsto \xi \mapsto y$ acts like $\zeta^{\frac{2}{3}}$ by Corollary 5. Thus,

$$q^{-1}\langle x \rangle^{-2} \sim E^{-1}\zeta(E^{-\frac{1}{2}}\zeta^{\frac{2}{3}})^{-2} \sim \zeta^{-2}$$

with each ζ derivative gaining one more power of decay in ζ .

In the remaining case $\zeta \leq -1$ one first calculates, on the one hand,

$$\begin{aligned} q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} &= \frac{5}{16\zeta^2} + \frac{1}{4} \frac{\ddot{Q}_0}{Q_0} - \frac{1}{8} \frac{\dot{Q}_0}{\zeta Q_0} - \frac{3}{16} \left(\frac{\dot{Q}_0}{Q_0} \right)^2 \\ &= \frac{5}{16\zeta^2} + q^{-1} \left[\frac{1}{4} \frac{V_0''}{Q_0} - \frac{5}{16} \left(\frac{V_0'}{Q_0} \right)^2 \right] \end{aligned}$$

where $' = \frac{d}{dx}$. Thus, from (3.22),

$$\begin{aligned} \tilde{V} &= \frac{1}{4} q^{-1} \langle x \rangle^{-2} - q^{-\frac{1}{4}} \frac{d^2 q^{\frac{1}{4}}}{d\zeta^2} \\ &= -\frac{5}{16\zeta^2} + q^{-1} \left[\frac{1}{4} \langle x \rangle^{-2} - \frac{1}{4} \frac{V_0''}{Q_0} + \frac{5}{16} \left(\frac{V_0'}{Q_0} \right)^2 \right] \\ &= -\frac{5}{16\zeta^2} + q^{-1} \left[E\beta_0(x, E) + \langle x \rangle^{-3} \beta_1(x, E) \right] \end{aligned}$$

where we have set⁴

$$(3.23) \quad \beta_0(x, E) := E^{-1} \left[\frac{1}{4} \left(\frac{V_0''}{V_0} - \frac{V_0''}{Q_0} \right) + \frac{5}{16} \left(\left(\frac{V_0'}{Q_0} \right)^2 - \left(\frac{V_0'}{V_0} \right)^2 \right) \right]$$

$$(3.24) \quad \beta_1(x, E) := \langle x \rangle^3 \left[\frac{1}{4} \langle x \rangle^{-2} - \frac{1}{4} \frac{V_0''}{V_0} + \frac{5}{16} \left(\frac{V_0'}{V_0} \right)^2 \right]$$

As already noted in Section 2, the x^{-2} terms inside the brackets in (3.24) cancel so that the leading order is x^{-3} . In fact, $|\partial_x^\ell \beta_1(x, E)| \leq C_\ell \langle x \rangle^{-\ell}$ in view of our assumptions on V , see Theorem 1. As for β_0 , we note that in the range $\zeta \leq -1$, one has $Q_0 \sim V_0$. Since $Q_0 = V_0 - E$, this implies that the expression in brackets in (3.23) is $\lesssim E$ together with the natural derivative bounds. The bounds on

$$q = (V_0 - E)|\zeta|^{-1} \sim \langle x \rangle^{-2} |\zeta|^{-1}, \quad |\zeta| \sim |\log(E\langle x \rangle^2)|^{\frac{2}{3}}$$

follow from (3.18) and we are done. \square

In Lemma 7 the modification of V to V_0 only played a role in the regime $\zeta \ll -1$ which is the same as $x < \varepsilon x_1(E)$. This is natural, since we know from Section 2 that this modification really comes from the $E = 0$ case which corresponds to $x_1 = +\infty$. We will see this mechanism at work in the following section, too.

⁴ β_1 does not depend on E , but this makes no difference.

4. SOLVING THE PERTURBED AIRY EQUATION

This section is devoted to solving (3.3), at least in the asymptotic sense relative to \hbar . We shall use the notations and results of the previous section. For the properties of the Airy functions Ai, Bi listed below we refer the reader to Chapter 11 of [19].

Proposition 8. *Let $\hbar_0 > 0$ be small. A fundamental system of solutions to (3.3) in the range $\zeta \leq 0$ is given by*

$$\begin{aligned}\phi_1(\zeta, E, \hbar) &= \text{Ai}(\tau)[1 + \hbar a_1(\zeta, E, \hbar)] \\ \phi_2(\zeta, E, \hbar) &= \text{Bi}(\tau)[1 + \hbar a_2(\zeta, E, \hbar)]\end{aligned}$$

with $\tau := -\hbar^{-\frac{2}{3}}\zeta$. Here a_1, a_2 are smooth, real-valued, and they satisfy the bounds, for all $k \geq 0$ and $j = 1, 2$, and with $\zeta_0 := \zeta(0, E)$,

$$(4.1) \quad \begin{aligned}|\partial_E^k a_j(\zeta, E, \hbar)| &\lesssim E^{-k} \min[\hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}}, 1] \\ |\partial_E^k \partial_\zeta a_j(\zeta, E, \hbar)| &\lesssim E^{-k} \left[\hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \right]\end{aligned}$$

uniformly in the parameters $0 < \hbar < \hbar_0$, $0 < E < E_0$.

Proof. Let $\phi_{1,0}(\zeta, \hbar) := \text{Ai}(\tau)$ and $\phi_{2,0}(\zeta, \hbar) := \text{Bi}(\tau)$. We seek a basis of the form

$$\phi_j(\zeta) = \phi(\zeta, \hbar, E) = \phi_{j,0}(\zeta, \hbar)(1 + \hbar a_j(\zeta, \hbar, E))$$

for $\zeta \leq 0$. This representation is meaningless for $\zeta > 0$ since $\phi_{j,0}$ have real zeros there. On the other hand, on $\zeta \leq 0$ they do not vanish. We obtain the equation

$$(4.2) \quad (\phi_{j,0}^2 \dot{a}_j)' = -\frac{1}{\hbar} \tilde{V} \phi_{j,0}^2 (1 + \hbar a_j)$$

for $j = 1, 2$ where $\dot{} = \partial_\zeta$. A solution of (4.2) on $\zeta \leq 0$ is given by, with $a_2(\zeta) = a_2(\zeta, \hbar, E)$,

$$(4.3) \quad \begin{aligned}a_2(\zeta) &:= -\frac{1}{\hbar} \int_\zeta^0 \phi_{2,0}^2(\eta, \hbar) \int_\zeta^\eta \phi_{2,0}^{-2}(\tilde{\eta}, \hbar) d\tilde{\eta} \tilde{V}(\eta, E) (1 + \hbar a_2(\eta)) d\eta \\ &= -\hbar^{\frac{1}{3}} \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] \tilde{V}(-\hbar^{\frac{2}{3}}u, E) (1 + \hbar a_2(\hbar^{\frac{2}{3}}u)) du\end{aligned}$$

This solution is unique with the property that $a_2(0) = \dot{a}_2(0) = 0$. Recall the asymptotic behavior, see [19],

$$\begin{aligned}\text{Bi}(x) &= \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{\frac{2}{3}x^{\frac{3}{2}}} [1 + O(x^{-\frac{3}{2}})] \quad \text{as } x \rightarrow \infty \\ \text{Bi}(x) &\geq \text{Bi}(0) > 0 \quad \forall x \geq 0 \\ \text{Ai}(x) &= \frac{1}{2} \pi^{-\frac{1}{2}} x^{-\frac{1}{4}} e^{-\frac{2}{3}x^{\frac{3}{2}}} [1 + O(x^{-\frac{3}{2}})] \quad \text{as } x \rightarrow \infty \\ \text{Ai}(x) &> 0 \quad \forall x \geq 0\end{aligned}$$

Also note the useful fact, valid for any $0 \leq x_0 < x_1$,

$$(4.4) \quad \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy = \pi^{-1} \left(\frac{\text{Ai}(x_0)}{\text{Bi}(x_0)} - \frac{\text{Ai}(x_1)}{\text{Bi}(x_1)} \right)$$

which implies that

$$\left| \text{Bi}^2(x_0) \int_{x_0}^{x_1} \text{Bi}^{-2}(y) dy \right| \lesssim \langle x_0 \rangle^{-\frac{1}{2}}$$

The leading term in (4.3), i.e.,

$$a_{2,0}(\zeta, E, \hbar) := -\hbar^{\frac{1}{3}} \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \left[\int_u^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^{-2}(v) dv \right] \tilde{V}(-\hbar^{\frac{2}{3}}u, E) du$$

therefore satisfies the bound (dropping E, \hbar from $a_{2,0}$ for simplicity)

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{3}} \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \langle u \rangle^{-\frac{1}{2}} |\tilde{V}(-\hbar^{\frac{2}{3}}u, E)| du$$

We now use the estimates from Lemma 7 to bound the right-hand side. If $-1 \leq \zeta \leq 0$, then this yields

$$(4.5) \quad |a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{3}} \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \langle u \rangle^{-\frac{1}{2}} du \lesssim \hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}}$$

On the other hand, if $\zeta_0 := \zeta(0, E) \leq \zeta \leq -1$, then we obtain

$$(4.6) \quad |a_{2,0}(\zeta)| \lesssim \hbar^{\frac{1}{3}} \int_0^{\hbar^{-\frac{2}{3}}} \langle u \rangle^{-\frac{1}{2}} du$$

$$(4.7) \quad + \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} \left[(\hbar^{\frac{2}{3}}u)^{-2} + q^{-1}(-\hbar^{\frac{2}{3}}u)(E + \langle z \rangle^{-3}) \right] du$$

The variable z appearing in (4.7) is tied to the integration variable u via $-\hbar^{\frac{2}{3}}u = \zeta(z, E)$, see Lemma 3. The integral in (4.6) and the first term inside the brackets in (4.7) contribute

$$\hbar^{\frac{1}{3}} \int_0^{\hbar^{-\frac{2}{3}}} \langle u \rangle^{-\frac{1}{2}} du + \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} (\hbar^{\frac{2}{3}}u)^{-2} du \lesssim 1$$

Next, with $\zeta(x_2, E) = -1$, and $v = \zeta(z, E)$, $dv = \sqrt{q}dz$,

$$\begin{aligned} \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} q^{-1}(\hbar^{\frac{2}{3}}u) E du &= E \int_1^{-\zeta} v^{-\frac{1}{2}} q^{-1}(v) dv \\ &= E \int_x^{x_2} Q_0^{-\frac{1}{2}}(z, E) dz \lesssim E \int_x^{x_1} z dz \\ &\lesssim E x_1(E)^2 \lesssim 1 \end{aligned}$$

Finally, using that $\frac{dv}{dz} = \sqrt{q}$ once again one obtains

$$\begin{aligned} \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} q^{-1}(\hbar^{\frac{2}{3}}u) z^{-3} du &= \int_1^{-\zeta} v^{-\frac{1}{2}} q^{-1}(v) \langle z \rangle^{-3} dv \\ &= \int_x^{x_2} Q_0^{-\frac{1}{2}}(z, E) \langle z \rangle^{-3} dz \\ &\lesssim \int_x^{x_1} \langle z \rangle^{-2} dz \lesssim 1 \end{aligned}$$

In summary⁵,

$$|a_{2,0}(\zeta, E, \hbar)| \lesssim \min(1, \hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}})$$

⁵Had we used V instead of V_0 in our definition of ζ , then we would be losing a factor of $\log E$ at this point. Indeed, for the case of V we would need to replace $E + \langle z \rangle^{-3}$ by the strictly weaker $\langle z \rangle^{-2}$ in (4.7) which then leads to the logarithmically divergent integral $\int_x^{x_1} \langle z \rangle^{-1} dz$.

uniformly in $\zeta \in [\zeta_0, 0]$, $0 < E < E_0$, and $0 < \hbar < \hbar_0$. Due to the linear nature of (4.3), a contraction argument now yields the same bound for a_2 ; in fact, due to the derivative bounds of Lemma 7, we obtain the more general estimate

$$|\partial_E^k a_2(\zeta, E, \hbar)| \leq C_k E^{-k} \min(1, \hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}}) \quad \forall k \geq 0$$

uniformly in the parameters. As for the first derivative in ζ , observe that

$$(4.8) \quad \dot{a}_2(\zeta) = \frac{\hbar^{-1}}{\phi_{2,0}^2(\zeta, \hbar)} \int_{\zeta}^0 \phi_{2,0}^2(\eta, \hbar) \tilde{V}(\eta, E) (1 + \hbar a_2(\eta)) d\eta$$

whence, for all $-1 \leq \zeta \leq 0$,

$$\begin{aligned} |\dot{a}_2(\zeta, E, \hbar)| &\lesssim \hbar^{-1} \phi_{2,0}^{-2}(\zeta, \hbar) \int_{\zeta}^0 \phi_{2,0}^2(\eta, \hbar) |\tilde{V}(\eta, E)| d\eta \\ &\lesssim \hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}} \zeta) \int_0^{-\hbar^{-\frac{2}{3}} \zeta} \text{Bi}^2(u) du \\ &\lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}} e^{-\frac{4}{3} \hbar^{-1} |\zeta|^{\frac{3}{2}}} \int_0^{-\hbar^{-\frac{2}{3}} \zeta} \langle u \rangle^{-\frac{1}{2}} e^{\frac{4}{3} u^{\frac{3}{2}}} du \\ &\lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}} e^{-\frac{4}{3} \hbar^{-1} |\zeta|^{\frac{3}{2}}} \int_0^{\hbar^{-1} |\zeta|^{\frac{3}{2}}} \langle v \rangle^{-\frac{1}{3}} |v|^{-\frac{1}{3}} e^{\frac{4}{3} v} dv \\ &\lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{\frac{1}{2}} e^{-\frac{4}{3} \hbar^{-1} |\zeta|^{\frac{3}{2}}} \langle \hbar^{-1} |\zeta|^{\frac{3}{2}} \rangle^{-\frac{2}{3}} e^{\frac{4}{3} \hbar^{-1} |\zeta|^{\frac{3}{2}}} \\ &\lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}} \zeta \rangle^{-\frac{1}{2}} \end{aligned}$$

If $\zeta_0 \leq \zeta \leq -1$, then

$$(4.9) \quad \begin{aligned} |\dot{a}_2(\zeta, E, \hbar)| &\lesssim \hbar^{-1} \phi_{2,0}^{-2}(\zeta, \hbar) \int_{\zeta}^0 \phi_{2,0}^2(\eta, \hbar) |\tilde{V}(\eta, E)| d\eta \\ &\lesssim \hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}} \zeta) \int_0^{\hbar^{-\frac{2}{3}} \zeta} \text{Bi}^2(u) du \end{aligned}$$

$$(4.10) \quad + \hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}} \zeta) \int_{\hbar^{-\frac{2}{3}} \zeta}^{-\hbar^{-\frac{2}{3}} \zeta} \text{Bi}^2(u) \left[(\hbar^{\frac{2}{3}} u)^{-2} + \frac{E + \langle z \rangle^{-3}}{q(-\hbar^{\frac{2}{3}} u)} \right] du$$

where z has the same meaning as in (4.7). First, note that (4.9) is rapidly (in fact, super exponentially) decreasing as $|\zeta|$ increases: $|(4.9)| \lesssim e^{-|\zeta|^{\frac{3}{2}}}$. Second, we bound the first part of (4.10) by

$$\begin{aligned} &\hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}} \zeta) \int_{\hbar^{-\frac{2}{3}} \zeta}^{-\hbar^{-\frac{2}{3}} \zeta} \text{Bi}^2(u) (\hbar^{\frac{2}{3}} u)^{-2} du \\ &\lesssim \hbar^{-\frac{5}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}} \zeta) \int_{\hbar^{-\frac{2}{3}} \zeta}^{-\hbar^{-\frac{2}{3}} \zeta} u^{-\frac{5}{2}} e^{\frac{4}{3} u^{\frac{3}{2}}} du \\ &\lesssim \hbar^{-\frac{5}{3}} (\hbar^{-\frac{2}{3}} |\zeta|)^{-\frac{5}{2}} \lesssim |\zeta|^{-\frac{5}{2}} \end{aligned}$$

The contribution to (4.10) involving E is

$$(4.11) \quad \begin{aligned} & \hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\zeta) \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^2(u)(q(\hbar^{\frac{2}{3}}u))^{-1} E du \\ & \lesssim E \hbar^{-\frac{1}{3}} (\hbar^{-\frac{2}{3}}|\zeta|)^{\frac{1}{2}} e^{-\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} e^{\frac{4}{3}u^{\frac{3}{2}}} (q(\hbar^{\frac{2}{3}}u))^{-1} du \end{aligned}$$

$$(4.12) \quad \lesssim E \hbar^{-1} |\zeta|^{\frac{1}{2}} e^{-\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \int_1^{-\zeta} z e^{\frac{4}{3\hbar}v^{\frac{3}{2}}} dz \lesssim |\zeta|^{\frac{1}{2}} E \langle x \rangle^2$$

To pass from (4.11) to (4.12), we substituted $u = \hbar^{-\frac{2}{3}}v$ and then changed variables $dv = \sqrt{q(-v)} dz$ followed by $vq(-v) = Q_0(z)$; in (4.12) the relation between v and z , as well as ζ and x , is given by (3.2). i.e., $v = \zeta(z, E)$, $\zeta = \zeta(x, E)$. To pass to the final inequality in (4.12) we integrate by parts so as to gain a factor of \hbar :

$$\begin{aligned} \int_1^{-\zeta} z e^{\frac{4}{3\hbar}v^{\frac{3}{2}}} dz & \lesssim \int_x^{x_2} z e^{\frac{2}{\hbar} \int_z^{x_1} \sqrt{Q_0(\eta)} d\eta} dz \\ & \lesssim \hbar \langle x \rangle^2 e^{\frac{2}{\hbar} \int_x^{x_1} \sqrt{Q_0(\eta)} d\eta} = \hbar \langle x \rangle^2 e^{\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \end{aligned}$$

where $\zeta(x_2, \hbar) = -1$. Finally, we turn the contribution of $\langle z \rangle^{-3}$ in (4.10). Using the same conventions regarding the relation between the variables this contribution is of the form

$$\begin{aligned} & \hbar^{-\frac{1}{3}} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\zeta) \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^2(u)(q(\hbar^{\frac{2}{3}}u))^{-1} \langle z \rangle^{-3} du \\ & \lesssim \hbar^{-\frac{1}{3}} (\hbar^{-\frac{2}{3}}|\zeta|)^{\frac{1}{2}} e^{-\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \int_{\hbar^{-\frac{2}{3}}}^{-\hbar^{-\frac{2}{3}}\zeta} u^{-\frac{1}{2}} e^{\frac{4}{3}u^{\frac{3}{2}}} (q(\hbar^{\frac{2}{3}}u))^{-1} \langle z \rangle^{-3} du \\ & \lesssim \hbar^{-1} |\zeta|^{\frac{1}{2}} e^{-\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \int_1^{-\zeta} \langle z \rangle^{-2} e^{\frac{4}{3\hbar}v^{\frac{3}{2}}} dz \lesssim |\zeta|^{\frac{1}{2}} \langle x \rangle^{-2} \end{aligned}$$

The final inequality here is based on the same kind of integration by parts as before:

$$\begin{aligned} \int_1^{-\zeta} \langle z \rangle^{-2} e^{\frac{4}{3\hbar}v^{\frac{3}{2}}} dz & \lesssim \int_x^{x_2} \langle z \rangle^{-2} e^{\frac{2}{\hbar} \int_z^{x_1} \sqrt{Q_0(\eta)} d\eta} dz \\ & \lesssim \hbar \langle x \rangle^{-2} e^{\frac{2}{\hbar} \int_x^{x_1} \sqrt{Q_0(\eta)} d\eta} = \hbar \langle x \rangle^{-2} e^{\frac{4}{3\hbar}|\zeta|^{\frac{3}{2}}} \end{aligned}$$

with x_2 as above. In conclusion, we estimate the contributions of (4.9) and (4.10) by

$$(4.13) \quad |\dot{a}_2(\zeta, E, \hbar)| \lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \leq \zeta \leq -1]}$$

as claimed.

Next, we turn to $\phi_1(\zeta, E)$ (dropping \hbar for simplicity). As usual we make the reduction ansatz

$$\phi_1(\zeta, E) = g(\zeta, E) \phi_2(\zeta, E)$$

which leads to the equation $(\phi_2^2 \dot{g})' = 0$. At this point it is convenient to extend the solutions ϕ_2 , which are originally defined on the interval $\zeta(0, E) \leq \zeta \leq 0$, to all of $\zeta \leq 0$. This is done in such a way that the bounds (4.1) remain valid for $\zeta \leq \zeta_0$

without, however, making any reference to the ODE (3.3) for those ζ . We can now solve for g in the form

$$\phi_1(\zeta, E) = \pi \hbar^{-\frac{2}{3}} \phi_2(\zeta, E) \int_{-\infty}^{\zeta} \phi_2(\eta, E)^{-2} d\eta$$

Inserting our representation of ϕ_2 into this formula yields

$$\phi_1(\zeta, E) = \pi \hbar^{-\frac{2}{3}} \text{Bi}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\eta)[1 + \hbar a_2(\eta, E)]^{-2} d\eta$$

First, we note that from (4.4),

$$\pi \hbar^{-\frac{2}{3}} \text{Bi}(-\hbar^{-\frac{2}{3}}\zeta) \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\eta) d\eta = \text{Ai}(-\hbar^{-\frac{2}{3}}\zeta)$$

Second, $[1 + \hbar a_2]^{-2} = 1 + \hbar \tilde{a}_2$ where \tilde{a}_2 satisfies the same bounds as a_2 (since $|a_2| \lesssim 1$). Thus, inspection of our formula for ϕ_1 reveals that $a_1 = \pi(a_2 + \tilde{a}_1)$ where

$$\begin{aligned} \tilde{a}_1(\zeta) &:= \hbar^{-\frac{2}{3}} \frac{\text{Bi}}{\text{Ai}}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \int_{-\infty}^{\zeta} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\eta) \tilde{a}_2(\eta, E) d\eta \\ &= \frac{\text{Bi}}{\text{Ai}}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{2}{3}}\eta, E) d\eta \end{aligned}$$

Furthermore, from (4.4),

$$\begin{aligned} \pi \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{2}{3}}\eta, E) d\eta &= - \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \tilde{a}_2(-\hbar^{\frac{2}{3}}\eta, E) d\left[\frac{\text{Ai}}{\text{Bi}}(\eta)\right] \\ (4.14) \quad &= \frac{\text{Ai}}{\text{Bi}}(-\hbar^{-\frac{2}{3}}\zeta) \tilde{a}_2(\zeta, E) - \hbar^{\frac{2}{3}} \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\eta) (\partial_1 \tilde{a}_2)(-\hbar^{\frac{2}{3}}\eta, E) d\eta \end{aligned}$$

where ∂_1 refers to the derivative in the first variable. The first term in (4.14) makes an admissible contribution to a_1 whereas the second one is controlled as follows:

$$\begin{aligned} &\hbar^{\frac{2}{3}} \frac{\text{Bi}}{\text{Ai}}(-\hbar^{-\frac{2}{3}}\zeta) \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\eta) |(\partial_1 \tilde{a}_2)(-\hbar^{\frac{2}{3}}\eta, E)| d\eta \\ &\lesssim \hbar^{\frac{2}{3}} e^{\frac{4}{3\hbar}\langle\zeta\rangle^{\frac{3}{2}}} \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} e^{-\frac{4}{3\hbar}\langle\eta\rangle^{\frac{3}{2}}} \left[\hbar^{-\frac{1}{3}} \langle\eta\rangle^{-\frac{1}{2}} \chi_{[-1 \leq \hbar^{\frac{2}{3}}\eta \leq 0]} + |\hbar^{\frac{2}{3}}\eta|^{\frac{1}{2}} \chi_{[\leq \hbar^{\frac{2}{3}}\eta \leq -1]} \right] d\eta \\ &\lesssim \hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-1} \chi_{[-1 \leq \zeta \leq 0]} + \hbar \chi_{[\zeta_0 \leq \zeta \leq -1]} \lesssim \hbar^{\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + \chi_{[\zeta_0 \leq \zeta \leq -1]} \end{aligned}$$

as desired. For the derivative in ζ ,

$$\begin{aligned} \partial_{\zeta} \tilde{a}_1(\zeta) &= -\pi \hbar^{-\frac{2}{3}} \text{Ai}^{-2}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{2}{3}}\eta, E) d\eta \\ &\quad + \hbar^{-\frac{2}{3}} (\text{AiBi})^{-1}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \tilde{a}_2(\zeta, E) \\ &\quad + \hbar \frac{\text{Bi}}{\text{Ai}}(-\hbar^{-\frac{2}{3}}\zeta) \dot{a}_2(\zeta, E) \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{2}{3}}\eta, E) d\eta \end{aligned}$$

Using (4.14) we remove the dangerous $\hbar^{-\frac{2}{3}}$ terms from the first two lines here whence

$$(4.15) \quad \begin{aligned} \partial_\zeta \tilde{a}_1(\zeta) &= \text{Ai}^{-2}(-\hbar^{-\frac{2}{3}}\zeta)[1 + \hbar a_2(\zeta, E)] \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \frac{\text{Ai}}{\text{Bi}}(\eta)(\partial_1 \tilde{a}_2)(-\hbar^{\frac{2}{3}}\eta, E) d\eta \\ &+ \hbar \frac{\text{Bi}}{\text{Ai}}(-\hbar^{-\frac{2}{3}}\zeta) \dot{a}_2(\zeta, E) \int_{-\hbar^{-\frac{2}{3}}\zeta}^{\infty} \text{Bi}^{-2}(\eta) \tilde{a}_2(\hbar^{\frac{2}{3}}\eta, E) d\eta \end{aligned}$$

The contribution by the first integral here is treated as the integral in (4.14) and is bounded by

$$\begin{aligned} &\lesssim \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \left[\hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-1} \chi_{[-1 \leq \zeta \leq 0]} + \hbar^{\frac{1}{3}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \right] \\ &\sim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \end{aligned}$$

which is exactly as needed. Finally, the contribution of (4.15) is bounded by

$$\begin{aligned} &\lesssim \hbar \left[\chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \right] \\ &\sim \hbar \langle \zeta \rangle^{\frac{1}{2}} \lesssim \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \chi_{[-1 \leq \zeta \leq 0]} + |\zeta|^{\frac{1}{2}} \chi_{[\zeta_0 \leq \zeta \leq -1]} \end{aligned}$$

and we are done with the $k = 0$ case of (4.1) for a_1 . However, since E enters into a_1 only through a_2, \tilde{a}_2 which do satisfy (4.1) for all $k \geq 0$, we see that the previous estimates carry over unchanged and provide the stated estimates for $\partial_E^k a_1(\zeta, E)$ and $\partial_E^k \partial_\zeta a_1(\zeta, E)$. \square

We remark that the method employed in the previous proof does not extend easily to derivatives $\partial_\zeta^\ell a_j$ with $\ell \geq 2$ (that is, without losing excessive powers of \hbar^{-1}). In principle, it is possible to treat $\ell = 2$ by a similar method, but instead of a sharp cut-off at $\zeta = 0$ one needs to use a smooth cut-off function in (4.3). However, the calculations are quite involved and it is not clear how to extend this approach systematically to higher ℓ (the same comment applies to Proposition 9 below). On the other hand, for the purposes of Theorem 1, as well as for those of [23] and [24], it suffices to treat the first derivative in ζ (however, we do need many derivatives relative to E). Next, we turn to $\zeta \geq 0$ which requires an oscillatory basis.

Proposition 9. *Let $\hbar_0 > 0$ be small. In the range $\zeta \geq 0$ a basis of solutions to (3.3) is given by*

$$\begin{aligned} \psi_1(\zeta, E; \hbar) &= (\text{Ai}(\tau) + i\text{Bi}(\tau))[1 + \hbar b_1(\zeta, E; \hbar)] \\ \psi_2(\zeta, E; \hbar) &= (\text{Ai}(\tau) - i\text{Bi}(\tau))[1 + \hbar b_2(\zeta, E; \hbar)] \end{aligned}$$

with $\tau := -\hbar^{-\frac{2}{3}}\zeta$ and where b_1, b_2 are smooth, complex-valued, and satisfy the bounds for all $k \geq 0$, and $j = 1, 2$

$$(4.16) \quad \begin{aligned} |\partial_E^k b_j(\zeta, E; \hbar)| &\leq C_k E^{-k} \langle \zeta \rangle^{-\frac{3}{2}} \\ |\partial_\zeta \partial_E^k b_j(\zeta, E)| &\leq C_k E^{-k} \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2} \end{aligned}$$

uniformly in the parameters $0 < \hbar < \hbar_0$, $0 < E < E_0$, $\zeta \geq 0$.

Proof. Let $\psi_{1,0}(\zeta; \hbar) := (\text{Ai} + i\text{Bi})(\tau)$ and $\psi_{2,0}(\zeta; \hbar) := (\text{Ai} - i\text{Bi})(\tau)$. We seek a basis of the form (dropping \hbar as an independent variable from the notation)

$$\psi_j(\zeta) = \psi_j(\zeta, E) = \psi_{j,0}(\zeta)(1 + \hbar b_j(\zeta, E))$$

for $\zeta \geq 0$. This representation is meaningful since Ai and Bi have no common zeros (as their Wronskian does not vanish). We obtain the equation

$$(4.17) \quad (\psi_{j,0}^2 \dot{b}_j) = -\frac{1}{\hbar} \tilde{V} \psi_{j,0}^2 (1 + \hbar b_j)$$

for $j = 1, 2$, c.f. (4.2). A solution of (4.17) on $\zeta \geq 0$ is given by, with $b_j(\zeta) = b_j(\zeta, E)$,

$$(4.18) \quad b_j(\zeta) := \frac{-1}{\hbar} \int_{\zeta}^{\infty} \psi_{j,0}^2(\eta) \int_{\zeta}^{\eta} \psi_{j,0}^{-2}(\tilde{\eta}) d\tilde{\eta} \tilde{V}(\eta, E) (1 + \hbar b_j(\eta)) d\eta$$

Recall the asymptotic behavior, see [19],

$$(4.19) \quad \text{Ai}(-x) \pm i\text{Bi}(-x) = \frac{1}{\pi^{\frac{1}{2}} x^{\frac{1}{4}}} e^{\mp i(\xi - \frac{\pi}{4})} (1 + O(\xi^{-1}))$$

as $x \rightarrow \infty$. Here $\xi = \frac{2}{3}x^{\frac{3}{2}}$ and the $O(\cdot)$ term is complex-valued and exhibits symbol behavior:

$$\partial_{\xi}^k O(\xi^{-1}) = O(\xi^{-1-k}) \quad \forall k \geq 0$$

Therefore, for any $0 > x_0 > x_1$,

$$\left| (\text{Ai} + i\text{Bi})^2(x_1) \int_{x_0}^{x_1} (\text{Ai} + i\text{Bi})^{-2}(y) dy \right| \lesssim \langle x_1 \rangle^{-\frac{1}{2}}$$

The leading term in (4.18), i.e.,

$$b_{1,0}(\zeta, \hbar, E) := \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}\zeta}^{\infty} (\text{Ai} + i\text{Bi})^2(-u) \left[\int_{\hbar^{-\frac{2}{3}}\zeta}^u (\text{Ai} + i\text{Bi})^{-2}(-v) dv \right] \tilde{V}(-\hbar^{\frac{2}{3}}u, E) du$$

therefore satisfies the bound, see Lemma 7,

$$\begin{aligned} |b_{1,0}(\zeta)| &\lesssim \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}\zeta}^{\infty} \langle u \rangle^{-\frac{1}{2}} |\tilde{V}(-\hbar^{\frac{2}{3}}u, E)| du \\ &\lesssim \hbar^{\frac{1}{3}} \int_{\hbar^{-\frac{2}{3}}\zeta}^{\infty} \langle u \rangle^{-\frac{1}{2}} \langle \hbar^{\frac{2}{3}}u \rangle^{-2} du \lesssim \langle \zeta \rangle^{-\frac{3}{2}} \end{aligned}$$

uniformly in $\zeta \geq 0$, $0 < E < E_0$, and $0 < \hbar < \hbar_0$. Due to the linear nature of (4.18), a contraction argument now yields the same bound for b_1 ; in fact, due to the derivative bounds of Lemma 7 relative to E , we obtain the more general estimate

$$|\partial_E^k b_j(\zeta, E)| \leq C_k E^{-k} \langle \zeta \rangle^{-\frac{3}{2}} \quad \forall k \geq 0$$

uniformly in the parameters for both $j = 1, 2$. As for the first derivative in ζ , observe that

$$(4.20) \quad \dot{b}_j(\zeta) = \frac{\hbar^{-1}}{\psi_{j,0}^2(\zeta)} \int_{\zeta}^{\infty} \psi_{j,0}^2(\eta) \tilde{V}(\eta, E) (1 + \hbar b_j(\eta)) d\eta$$

In order to exploit the cancellation in this integral, one integrates by parts once. To this end, write for $u \geq 0$,

$$\begin{aligned} (\text{Ai} + i\text{Bi})^2(-u) &= e^{\frac{4i}{3}\langle u \rangle^{\frac{3}{2}}} \omega(u), \quad |\omega(u)| \lesssim \langle u \rangle^{-\frac{1}{2}}, \quad |\omega'(u)| \lesssim \langle u \rangle^{-\frac{3}{2}}, \\ \psi_{1,0}^2(\zeta; \hbar) &= e^{\frac{4i}{3}\langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{3}{2}}} \omega(\hbar^{-\frac{2}{3}}\zeta) \end{aligned}$$

Since

$$\psi_{1,0}^2(\zeta; \hbar) d\zeta = \frac{1}{2i} \hbar^{\frac{2}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \omega(\hbar^{-\frac{2}{3}}\zeta) d \left[e^{\frac{4i}{3}\langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{3}{2}}} \right]$$

integration by parts yields

$$\begin{aligned} \dot{b}_1(\zeta) &= \frac{\hbar^{-\frac{1}{3}}}{2i\psi_{j,0}^2(\zeta)} \int_{\zeta}^{\infty} \langle \hbar^{-\frac{2}{3}}\eta \rangle^{-\frac{1}{2}} \omega(\hbar^{-\frac{2}{3}}\eta) \widetilde{V}(\eta, E) (1 + \hbar b_j(\eta)) d \left[e^{\frac{4i}{3} \langle \hbar^{-\frac{2}{3}}\eta \rangle^{\frac{3}{2}}} \right] \\ (4.21) \quad &= O(\hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}) - \end{aligned}$$

$$(4.22) \quad - \frac{\hbar^{\frac{2}{3}}}{2i\psi_{j,0}^2(\zeta)} \int_{\zeta}^{\infty} e^{\frac{4i}{3} \langle \hbar^{-\frac{2}{3}}\eta \rangle^{\frac{3}{2}}} O(\langle \hbar^{-\frac{2}{3}}\eta \rangle^{-1} \langle \eta \rangle^{-2}) \dot{b}_1(\eta) d\eta$$

The leading order here is given by (4.21); indeed, if we estimate the $\dot{b}_1(\eta)$ term in (4.22) by (4.21), then (4.22) $\lesssim \hbar \langle \zeta \rangle^{-4}$, which is much better than (4.21). The conclusion is that

$$|\partial_{\zeta} \partial_E^k b_j(\zeta, E)| \lesssim E^{-k} \hbar^{-\frac{1}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \langle \zeta \rangle^{-2}$$

as claimed. \square

5. THE PROOF OF THEOREM 1

Let $f_{\pm}(x, E; \hbar)$ be the Jost solutions of $P(x, \hbar D)$ from (1.1). For ease of notation, we shall first assume the symmetry $V(x) = V(-x)$ and later indicate how to treat the general case. Also, as usual, we drop \hbar from the arguments of functions. Then $f_-(x, E) = f_+(-x, E)$ so that the Wronskian of f_+, f_- is

$$W(E) = -2f_+(0, E)f'_+(0, E)$$

Next, from (4.19), and with $\zeta = \zeta(x, E)$ as in (3.2) and $T_+(E)$ as in (1.5),

$$f_+(x, E) = \sqrt{\pi} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} e^{i(\frac{T_+(E)}{\hbar} + \frac{\pi}{4})} q^{-\frac{1}{4}}(\zeta) \psi_2(\zeta, E)$$

This is obtained by matching the asymptotic behavior of f_+ with that of $\psi_2(\zeta)$ as $x \rightarrow \infty$ and we used the relation $w = q^{\frac{1}{4}} f$ from Lemma 3. We now connect ψ_2 to the basis $\phi_j(\zeta, E)$ of Proposition 8:

$$\psi_2(\zeta, E) = c_1(E)\phi_1(\zeta, E) + c_2(E)\phi_2(\zeta, E)$$

where

$$c_1(E) = \frac{W(\psi_2(\cdot, E), \phi_2(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}, \quad c_2(E) = -\frac{W(\psi_2(\cdot, E), \phi_1(\cdot, E))}{W(\phi_1(\cdot, E), \phi_2(\cdot, E))}$$

By Proposition 8,

$$W(\phi_1(\cdot, E), \phi_2(\cdot, E)) = -\hbar^{-\frac{2}{3}} W(\text{Ai}, \text{Bi}) + O(\hbar^{-\frac{2}{3}}) = -\pi^{-1} \hbar^{-\frac{2}{3}} (1 + O(\hbar))$$

where we evaluated the Wronskian on the left-hand side at $\zeta = 0$. Next, by Propositions 8 and 9,

$$\begin{aligned} W(\psi_2(\cdot, E), \phi_2(\cdot, E)) &= -\hbar^{-\frac{2}{3}} [(\text{Ai}(0) - i\text{Bi}(0))\text{Bi}'(0) \\ &\quad - (\text{Ai}'(0) - i\text{Bi}'(0))\text{Bi}(0) + O(\hbar)] \\ &= -\hbar^{-\frac{2}{3}} [W(\text{Ai}, \text{Bi}) + O(\hbar)] \\ (5.1) \quad W(\psi_1(\cdot, E), \phi_1(\cdot, E)) &= -\hbar^{-\frac{2}{3}} [(\text{Ai}(0) - i\text{Bi}(0))\text{Ai}'(0) \\ &\quad - (\text{Ai}'(0) - i\text{Bi}'(0))\text{Ai}(0) + O(\hbar)] \\ &= -\hbar^{-\frac{2}{3}} [iW(\text{Ai}, \text{Bi}) + O(\hbar)] \end{aligned}$$

so that

$$(5.2) \quad c_1(E) = 1 + O(\hbar), \quad c_2(E) = -i + O(\hbar)$$

where the $O(\cdot)$ terms satisfy $|\partial_E^k O(\hbar)| \leq C_k E^{-k}$. For the remainder of the proof, we set $\zeta_0 := \zeta(0, E)$. Then

$$\begin{aligned} f_+(0, E) &= \sqrt{\pi} e^{i(\frac{T_+(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} q^{-\frac{1}{4}}(\zeta_0) \psi_2(\zeta_0, E) \\ &= \sqrt{\pi} e^{i(\frac{T_+(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} q^{-\frac{1}{4}}(\zeta_0) [c_1(E) \phi_1(\zeta_0, E) + c_2(E) \phi_2(\zeta_0, E)] \\ f'_+(0, E) &= \sqrt{\pi} e^{i(\frac{T_+(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} \zeta'(0) q^{-\frac{1}{4}}(\zeta_0) [\psi'_2(\zeta_0, E) - \frac{1}{4} \frac{\dot{q}}{q}(\zeta_0) \psi_2(\zeta_0, E)] \\ &= \sqrt{\pi} e^{i(\frac{T_+(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} \zeta'(0) q^{-\frac{1}{4}}(\zeta_0) [c_1(E) \phi'_1(\zeta_0, E) + c_2(E) \phi'_2(\zeta_0, E) \\ &\quad - \frac{1}{4} \frac{\dot{q}}{q}(\zeta_0) (c_1(E) \phi_1(\zeta_0, E) + c_2(E) \phi_2(\zeta_0, E))] \end{aligned}$$

Recall from Lemma 3 that $\zeta' = q^{\frac{1}{2}}$. From $V'(0) = 0$ we obtain

$$\dot{q}(\zeta_0) = \frac{Q_0(0)}{\zeta_0^2} = -\frac{q(\zeta_0)}{\zeta_0}$$

and thus

$$\begin{aligned} f_+(0, E) f'_+(0, E) &= i \pi E^{\frac{1}{2}} e^{2i \frac{T_+(E)}{\hbar}} \hbar^{-\frac{1}{3}} [c_1(E) \phi_1(\zeta_0, E) + c_2(E) \phi_2(\zeta_0, E)] \times \\ &\quad \times [c_1(E) \phi'_1(\zeta_0, E) + c_2(E) \phi'_2(\zeta_0, E)] \\ &\quad + \frac{i}{4} \pi E^{\frac{1}{2}} e^{2i \frac{T_+(E)}{\hbar}} \hbar^{-\frac{1}{3}} \zeta_0^{-1} [c_1(E) \phi_1(\zeta_0, E) + c_2(E) \phi_2(\zeta_0, E)]^2 \end{aligned}$$

From Proposition 8,

$$\begin{aligned} \phi_1(\zeta_0, E) &= \text{Ai}(-\hbar^{-\frac{2}{3}} \zeta_0) (1 + O(\hbar)) \\ \phi_2(\zeta_0, E) &= \text{Bi}(-\hbar^{-\frac{2}{3}} \zeta_0) (1 + O(\hbar)) \\ \phi'_1(\zeta_0, E) &= -\hbar^{-\frac{2}{3}} \text{Ai}'(-\hbar^{-\frac{2}{3}} \zeta_0) (1 + O(\hbar)) + O(\hbar) |\zeta_0|^{\frac{1}{2}} \text{Ai}(-\hbar^{-\frac{2}{3}} \zeta_0) \\ \phi'_2(\zeta_0, E) &= -\hbar^{-\frac{2}{3}} \text{Bi}'(-\hbar^{-\frac{2}{3}} \zeta_0) (1 + O(\hbar)) + O(\hbar) |\zeta_0|^{\frac{1}{2}} \text{Bi}(-\hbar^{-\frac{2}{3}} \zeta_0) \end{aligned}$$

which implies via the standard asymptotics of the Airy functions that

$$\begin{aligned} \phi_1(\zeta_0, E) &= (4\pi)^{-\frac{1}{2}} (\hbar^{-\frac{2}{3}} |\zeta_0|)^{-\frac{1}{4}} e^{-\frac{2}{3} \hbar^{-1} |\zeta_0|^{\frac{3}{2}}} (1 + O(\hbar)) \\ \phi_2(\zeta_0, E) &= \pi^{-\frac{1}{2}} (\hbar^{-\frac{2}{3}} |\zeta_0|)^{-\frac{1}{4}} e^{\frac{2}{3} \hbar^{-1} |\zeta_0|^{\frac{3}{2}}} (1 + O(\hbar)) \\ \phi'_1(\zeta_0, E) &= \hbar^{-\frac{2}{3}} (4\pi)^{-\frac{1}{2}} (\hbar^{-\frac{2}{3}} |\zeta_0|)^{\frac{1}{4}} e^{-\frac{2}{3} \hbar^{-1} |\zeta_0|^{\frac{3}{2}}} (1 + O(\hbar)) \\ \phi'_2(\zeta_0, E) &= -\hbar^{-\frac{2}{3}} \pi^{-\frac{1}{2}} (\hbar^{-\frac{2}{3}} |\zeta_0|)^{\frac{1}{4}} e^{\frac{2}{3} \hbar^{-1} |\zeta_0|^{\frac{3}{2}}} (1 + O(\hbar)) \end{aligned}$$

Hence, using that $e^{-\hbar^{-1} |\zeta_0|^{\frac{3}{2}}} = O(\hbar)$ where $\partial_E^k O(\hbar) = O(E^{-k} \hbar)$, one concludes that

$$\begin{aligned} &\hbar^{-\frac{1}{3}} [c_1(E) \phi_1(\zeta_0, E) + c_2(E) \phi_2(\zeta_0, E)] \times \\ &\quad \times [c_1(E) \phi'_1(\zeta_0, E) + c_2(E) \phi'_2(\zeta_0, E)] \\ &= \pi^{-1} \hbar^{-1} e^{\frac{4}{3} \hbar^{-1} |\zeta_0|^{\frac{3}{2}}} (1 + O(\hbar)) \end{aligned}$$

as well as

$$\hbar^{-\frac{1}{3}}\zeta_0^{-1}[c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)]^2 = -\pi^{-1}|\zeta_0|^{-\frac{3}{2}}e^{\frac{4}{3}\hbar^{-1}|\zeta_0|^{\frac{3}{2}}}(1 + O(\hbar))$$

Since $T(E) = 2T_+(E)$ we finally arrive at

$$\begin{aligned} W(E) &= -2f_+(0, E)f'_+(0, E) = -2ie^{2i\frac{T_+(E)}{\hbar}}E^{\frac{1}{2}}\hbar^{-1}e^{\frac{4}{3}\hbar^{-1}|\zeta_0|^{\frac{3}{2}}}(1 + O(\hbar)) \\ (5.3) \quad &= -\frac{2i\sqrt{E}}{\hbar}e^{\hbar^{-1}(S(E)+iT(E))}(1 + O(\hbar)) \end{aligned}$$

We used here that

$$\frac{4}{3}|\zeta_0|^{\frac{3}{2}} = 2 \int_0^{x_1} \sqrt{V_0(\eta) - E} d\eta = S(E)$$

All the $O(\hbar)$ appearing above behave as required under differentiation with respect to E ; indeed, this is both due to the bounds of Propositions 8 and 9 as well as the aforementioned fact that

$$e^{-\frac{2}{3}\hbar^{-1}|\zeta_0|^{\frac{3}{2}}} = O(\hbar|\zeta_0|^{-\frac{3}{2}}) = O(\hbar)$$

has the required behavior since $|\zeta_0|^{-\frac{3}{2}} = O(|\log E|^{-\frac{3}{2}})$ as $E \rightarrow 0+$. In view of (1.9), (5.3) implies the sought after asymptotic relation for \mathbb{S}_{11} in Theorem 1, see (1.6). In order to find \mathbb{S}_{12} , and \mathbb{S}_{21} (i.e., the reflection coefficients), we need to also asymptotically evaluate the following Wronskians:

$$W(f_+(\cdot, E), \overline{f_-(\cdot, E)}) = W(\overline{f_+(\cdot, E)}, f_-(\cdot, E)) = -2\operatorname{Re}[f_+(0, E)\overline{f'_+(0, E)}].$$

Using the same notations as in the computation of $W(E)$, we obtain

$$\begin{aligned} -2\operatorname{Re}[f_+(0, E)\overline{f'_+(0, E)}] &= -2\operatorname{Re}\left\{\pi E^{\frac{1}{2}}\hbar^{-\frac{1}{3}}[c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)] \times \right. \\ &\quad \times [\overline{c_1(E)}\phi'_1(\zeta_0, E) + \overline{c_2(E)}\phi'_2(\zeta_0, E)]\left.\right\} \\ &\quad - \frac{\pi}{2}E^{\frac{1}{2}}\hbar^{-\frac{1}{3}}\zeta_0^{-1}|c_1(E)\phi_1(\zeta_0, E) + c_2(E)\phi_2(\zeta_0, E)|^2 \end{aligned}$$

Finally, evaluating this expression as above, we obtain

$$W(f_+(\cdot, E), \overline{f_-(\cdot, E)}) = -2\operatorname{Re}[f_+(0, E)\overline{f'_+(0, E)}] = \frac{2\sqrt{E}}{\hbar}e^{\frac{S(E)}{\hbar}}(1 + O(\hbar)).$$

Forming the ratio between this formula and the one for $W(E)$ yields the desired expression for $\mathbb{S}_{12} = \mathbb{S}_{21}$, see (1.6). Indeed,

$$r_-(E) = -\frac{W(\overline{f_+(\cdot, E)}, f_-(\cdot, E))}{W(f_+(\cdot, E), \overline{f_-(\cdot, E)})} = -ie^{-i\hbar^{-1}T(E)}(1 + O(\hbar))$$

where $O(\hbar)$ behaves like a symbol with respect to E , as usual. This concludes the proof of Theorem 1 in the symmetric case. If $V(x) \neq V(-x)$, then only minor changes are needed. Indeed, on $x \leq 0$ we can still use the *same bases* ϕ_j, ψ_j from Section 4 but with $\zeta = \zeta(-x, E)$. This is due to the fact that the difference between the left-hand and right-hand branches of V does not affect the estimates from Section 4 (since we are assuming inverse square decay at both ends and the constants μ_{\pm} have no effect on the leading order behavior). Let

$$\tilde{\zeta}_0(E)^{\frac{3}{2}} := \frac{3}{2} \int_{x_2(E)}^0 \sqrt{V_0(\eta) - E} d\eta$$

Thus, in addition to the expressions for $f_+(0, E)$ and $f'_+(0, E)$ from above we now also have

$$\begin{aligned} f_-(0, E) &= \sqrt{\pi} e^{i(\frac{T_-(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} q^{-\frac{1}{4}}(\tilde{\zeta}_0) [c_1(E) \phi_1(\tilde{\zeta}_0, E) + c_2(E) \phi_2(\tilde{\zeta}_0, E)] \\ f'_-(0, E) &= -\sqrt{\pi} e^{i(\frac{T_-(E)}{\hbar} + \frac{\pi}{4})} E^{\frac{1}{4}} \hbar^{-\frac{1}{6}} \zeta'(0) q^{-\frac{1}{4}}(\tilde{\zeta}_0) [c_1(E) \phi'_1(\tilde{\zeta}_0, E) + c_2(E) \phi'_2(\tilde{\zeta}_0, E)] \\ &\quad - \frac{1}{4} \frac{\dot{q}}{q}(\tilde{\zeta}_0) (c_1(E) \phi_1(\tilde{\zeta}_0, E) + c_2(E) \phi_2(\tilde{\zeta}_0, E)) \end{aligned}$$

Inserting these expressions into

$$W(E) = f_+(0, E) f'_-(0, E) - f'_+(0, E) f_-(0, E),$$

and using that

$$\frac{2}{3} [\zeta_0^{\frac{3}{2}} + \tilde{\zeta}_0^{\frac{3}{2}}] = \int_{x_2(E)}^{x_1(E)} \sqrt{V_0(\eta) - E} d\eta = S(E)$$

as well as $T(E) = T_+(E) + T_-(E)$, one again arrives at (5.3). The same comments apply to the off-diagonal terms of the scattering matrix and we are done. As for the very last claim of the theorem concerning $V_0 = V + \hbar^2 V_1$, simply note that the main calculations entering into the above proof only make use of the leading order part of V_1 , i.e., $\frac{1}{4}\langle x \rangle^{-2}$ whereas the cubic piece gets absorbed into the error term.

6. FROM SMALL TO LARGE ENERGIES

In this section, we present an extension of Theorem 1 to the case of large energies. More specifically, suppose V is as in Theorem 1 but with the following additional properties:

- $0 < V(x) \leq 1$ for all $x \in \mathbb{R}$, $V(0) = 1$, $V'(0) = 0$, $V''(0) = -1$
- $V'(x) < 0$ for all $x > 0$, $V'(x) > 0$ for all $x < 0$

Note that this is precisely the kind of barrier potential considered by Ramond [22] (but without any analyticity assumptions). For the purposes of this section we refer to it as a *simple barrier potential*. Even though Theorem 1 by design only considers *small* energies $0 < E < E_0$, it is natural to ask to what extent it remains correct as $E_0 \rightarrow 1$. As already remarked before, for energies $E > \varepsilon > 0$ there is no difference between V and V_0 as far as Theorem 1 is concerned. Indeed, switching from V to V_0 only affects the error term. Moreover, for the kind of V we are considering here, the theorem remains valid in any range $0 < E < 1 - \varepsilon$ with ε fixed. This is due to the fact that in this range there is a unique pair of turning points $x_2(E), x_1(E)$ as before. The action $S(E; \hbar)$ lies between two positive constants (depending on ε) and the previous proof goes through without changes. Somewhat more interesting and very relevant for later applications, cf. [23], [24], is the case where $\varepsilon = \hbar^\alpha$. The question is then how large $\alpha \geq 0$ can be allowed to be. First note that we can no longer expect the error term in (1.6) to be of the form $O(\hbar)$ in that case. Rather, it will need to be $O(\hbar^\delta)$ for some $\delta = \delta(\alpha) > 0$ and this condition will determine how large we can take α . It turns out that the range $0 \leq \alpha < 1$ is admissible here. In the following corollary, we use the notations introduced in Theorem 1.

Corollary 10. *Let V be a simple barrier potential. For every $0 < \alpha < 1$ there exists and $\hbar_0 = \hbar_0(\alpha)$ small such that for all $0 < \hbar < \hbar_0$ and $0 < E \leq 1 - \hbar^\alpha$*

$$(6.1) \quad \begin{aligned} \mathbb{S}_{11}(E; \hbar) &= e^{-\frac{1}{\hbar}(S(E; \hbar) + iT(E; \hbar))} (1 + \hbar(1 - E)^{-1} \sigma_{11}(E; \hbar)) \\ \mathbb{S}_{12}(E; \hbar) &= -ie^{-\frac{2i}{\hbar}T_+(E; \hbar)} (1 + \hbar(1 - E)^{-1} \sigma_{12}(E; \hbar)) \end{aligned}$$

and the correction terms satisfy the bounds

$$(6.2) \quad |\partial_E^k \sigma_{11}(E; \hbar)| + |\partial_E^k \sigma_{12}(E; \hbar)| \leq C_k \max(E^{-k}, (1 - E)^{-k}) \quad \forall k \geq 0,$$

with a constant C_k that only depends on k and V .

Proof. We will only sketch the proof as there is no point in repeating all the details of the proof of Theorem 1. In fact, inspection of the previous section shows that the main issue is to prove that Propositions 8 and 9 remain valid albeit with errors of the form $\hbar^{1-\alpha}$ rather than \hbar (we need to pay particular attention to the derivative ∂_ζ) when $E = 1 - \hbar^\alpha$. We will freely use the notation from Section 3 and 4. By the preceding comments, it will suffice to consider the range $1 - \varepsilon < E \leq 1 - \hbar^\alpha$. In fact, it will be enough to set $E = 1 - \hbar^\alpha$ so that $x_1(E) \sim \hbar^{\frac{\alpha}{2}}$. The range $0 < x < x_1(E)$ then corresponds to the region $-\hbar^{\frac{2\alpha}{3}} \lesssim \zeta \leq 0$. A simple calculation shows that $q \sim \hbar^{\frac{\alpha}{3}}$ in that range, as well as $|\tilde{V}| \lesssim \hbar^{-\frac{4\alpha}{3}}$ with the usual behavior under differentiation in E . In fact, for all $0 \leq x \leq x_1(E)$ we have

$$V(x) - E = - \int_x^{x_1} V'(y) dy \sim x_1^2 - x^2 \sim (x_1 - x)x_1$$

and thus

$$\zeta \sim -x_1^{\frac{1}{3}}(x_1 - x), \quad q = \frac{V - E}{-\zeta} \sim \frac{x_1(x_1 - x)}{x_1^{\frac{1}{3}}(x_1 - x)} = x_1^{\frac{2}{3}} \sim \hbar^{\frac{\alpha}{3}}$$

as claimed. Next, recall (3.22), viz.

$$(6.3) \quad \tilde{V} = \frac{1}{4}q^{-1}\langle x \rangle^{-2} + \frac{3}{16}q^{-2}\dot{q}^2 - \frac{1}{4}q^{-1}\ddot{q}$$

Since $\dot{q} = q^{-\frac{1}{2}}q'$ where $q' = \frac{dq}{dx} \sim x_1^{-\frac{1}{3}} \sim q^{-\frac{1}{2}}$, the second term here is of size

$$q^{-2}\dot{q}^2 \lesssim q^{-3}(q')^2 \lesssim q^{-4} \sim \hbar^{-\frac{4\alpha}{3}}$$

The other two terms are smaller whence $|\tilde{V}| \lesssim \hbar^{-\frac{4\alpha}{3}}$ as claimed. Turning to Proposition 8, we seek a basis of the form

$$\begin{aligned} \phi_1(\zeta, E, \hbar) &= \text{Ai}(\tau)[1 + \hbar^\delta a_1(\zeta, E, \hbar)] \\ \phi_2(\zeta, E, \hbar) &= \text{Bi}(\tau)[1 + \hbar^\delta a_2(\zeta, E, \hbar)] \end{aligned}$$

with $\delta := 1 - \alpha$. Proceeding as in the proof of Theorem 1, we arrive at the following analogue of (4.5)

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{4}{3}-\delta} \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \langle u \rangle^{-\frac{1}{2}} |\tilde{V}(-\hbar^{\frac{2}{3}}u, E)| du$$

which yields

$$|a_{2,0}(\zeta)| \lesssim \hbar^{\frac{4}{3}-\delta} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{\frac{1}{2}} \hbar^{-\frac{4\alpha}{3}} \lesssim \hbar^{1-\alpha-\delta}$$

This shows that with our choice of δ , we have

$$\sup_{\zeta(0,E) \leq \zeta \leq 0} |a_{2,0}(\zeta)| \lesssim 1$$

For the derivatives, the analogue of (4.8), viz.,

$$\dot{a}_2(\zeta) = \frac{\hbar^{-\delta}}{\phi_{2,0}^2(\zeta, \hbar)} \int_{\zeta}^0 \phi_{2,0}^2(\eta, \hbar) \tilde{V}(\eta, E) (1 + \hbar a_2(\eta)) d\eta$$

yields

$$\begin{aligned} |\dot{a}_2(\zeta, E, \hbar)| &\lesssim \hbar^{\frac{2}{3}-\delta} \text{Bi}^{-2}(-\hbar^{-\frac{2}{3}}\zeta) \int_0^{-\hbar^{-\frac{2}{3}}\zeta} \text{Bi}^2(u) \hbar^{-\frac{4\alpha}{3}} du \\ &\lesssim \hbar^{\frac{2}{3}-\delta-\frac{4\alpha}{3}} \langle \hbar^{-\frac{2}{3}}\zeta \rangle^{-\frac{1}{2}} \lesssim \hbar^{-\frac{2}{3}} \end{aligned}$$

where we again used that $\alpha < 1$ in the final step. An analogous estimate holds for ϕ_1 . We claim that these bounds are sufficient *provided* the same type of estimates hold for the analogue of Proposition 9 at $\zeta = 0$. Indeed, inspection of (5.1) shows that in that case

$$\begin{aligned} W(\psi_2(\cdot, E), \phi_2(\cdot, E)) &= -\hbar^{-\frac{2}{3}}[(\text{Ai}(0) - i\text{Bi}(0))\text{Bi}'(0) \\ &\quad - (\text{Ai}'(0) - i\text{Bi}'(0))\text{Bi}(0) + O(\hbar^\delta)] \\ &= -\hbar^{-\frac{2}{3}}[W(\text{Ai}, \text{Bi}) + O(\hbar^\delta)] \\ W(\psi_1(\cdot, E), \phi_1(\cdot, E)) &= -\hbar^{-\frac{2}{3}}[(\text{Ai}(0) - i\text{Bi}(0))\text{Ai}'(0) \\ &\quad - (\text{Ai}'(0) - i\text{Bi}'(0))\text{Ai}(0) + O(\hbar^\delta)] \\ &= -\hbar^{-\frac{2}{3}}[iW(\text{Ai}, \text{Bi}) + O(\hbar^\delta)] \end{aligned}$$

Note that there is an exact balance here between the $\hbar^{-\frac{2}{3}}$ coming from the derivatives of the main contributions and the losses stemming from \dot{a}_j, \dot{b}_j . Hence,

$$c_1(E) = 1 + O(\hbar^\delta), \quad c_2(E) = -i + O(\hbar^\delta)$$

as desired. Since $\hbar^{-1}|\zeta_0|^{\frac{3}{2}} \sim \hbar^{\alpha-1}$ and thus also

$$e^{-\hbar^{-1}|\zeta_0|^{\frac{3}{2}}} = O(\hbar^{1-\alpha}) = O(\hbar^\delta)$$

the reader will easily check that the remainder of the proof in Section 5 goes through.

It therefore remains to deal with the oscillatory regime. In analogy with Proposition 9 we seek a basis

$$\begin{aligned} \psi_1(\zeta, E; \hbar) &= (\text{Ai}(\tau) + i\text{Bi}(\tau))[1 + \hbar^\delta b_1(\zeta, E; \hbar)] \\ \psi_2(\zeta, E; \hbar) &= (\text{Ai}(\tau) - i\text{Bi}(\tau))[1 + \hbar^\delta b_2(\zeta, E; \hbar)] \end{aligned}$$

For this we need to understand \tilde{V} on $\zeta \geq 0$. First one checks that for all $x \geq x_1(E)$,

$$\zeta \sim \begin{cases} x_1^{\frac{1}{3}}(x - x_1) & x_1 \leq x \leq 2x_1 \\ x^{\frac{2}{3}} & 2x_1 \leq x \ll 1 \\ x^{\frac{2}{3}} & x \gtrsim 1 \end{cases}$$

and thus

$$q \sim \begin{cases} x^{\frac{2}{3}} & x_1 \leq x \ll 1 \\ x^{-\frac{2}{3}} & x \gtrsim 1 \end{cases}$$

Hence, (3.22) implies that

$$|\tilde{V}| \lesssim x^{-\frac{8}{3}} \chi_{[x_1 \leq x \leq 1]} + \zeta^{-2} \chi_{[x \geq 1]}$$

Going through the proof of Proposition 9 shows that

$$|b_j(0)| \lesssim \hbar^{1-\alpha-\delta} \lesssim 1, \quad |\dot{b}_j(0)| \lesssim \hbar^{\frac{2}{3}-\delta-\frac{4\alpha}{3}} \lesssim \hbar^{-\frac{2}{3}}$$

as desired. The derivatives relative to E are left to the reader. \square

Thus, the semi-classical approximation obtained in Theorem 1 breaks down precisely at $E = 1 - \hbar$. As is well-known, the Airy equation is no longer the correct approximating equation for energies close to the unique maximum $V(0) = 1$ of a simple barrier potential. In fact, there exists an analytic change of variables which reduces the Schrödinger equation with such energies to the Weber equation locally around the origin. Alternatively, Ramond [22] invokes micro-local methods and the Helffer-Sjöstrand normal form in that case.

APPENDIX A. A NORMAL FORM REDUCTION TO BESSEL'S EQUATION

In this section we sketch an alternative route for the asymptotic analysis of Section 4. It is based on Lemma 4 and reduces equation (3.1) to a Bessel equation rather than an Airy equation. However, we emphasize that these approaches are in fact quite related as the Airy functions are used to describe Bessel functions J_n and Y_n in the large $n = \hbar^{-1}$ asymptotics very much in the spirit of Section 4, see [19]. A possible advantage of working with the Bessel representation lies with the fact that they apply to all $x \in [\varepsilon E^{-\frac{1}{2}}, \infty)$ which is a region containing the turning point $x_1(E)$. On the other hand, since they cannot be used on the region $[0, \varepsilon E^{-\frac{1}{2}}]$, one is again faced with a connection problem as in Section 4. Moreover, we have found that using distinct changes of variables in these two regions leads to a number of complications as compared to the global action-based coordinates introduced in Lemma 3. For this, as well as other reasons, we ultimately found it technically advantageous to work with the Airy approximation directly, but we wish to sketch the Bessel method since it seems to be of independent interest. In this section, we shall use the notations of Lemma 4 and always work on $y \geq 1$ which transforms into $\xi \geq \xi_1(E)$, see (3.13). First, a preliminary technical lemma.

Lemma 11. *The function $\mu(\eta, E) := (\partial_\xi y(\eta, E))^2 (\partial_{yy} \xi)(y(\eta, E), E)$ satisfies*

$$(A.1) \quad |\partial_E^k \partial_\eta^j \mu(\eta, E)| \leq C_{kj} E^{-k} \eta^{-j-3}$$

and the positive smooth function

$$(A.2) \quad \Omega(\eta, E) := \exp \left(- \int_\eta^\infty \mu(t, E) dt \right)$$

satisfies $\frac{\Omega'}{\Omega} = \mu$ and $\Omega = 1 + O(\eta^{-2})$ (we write $' = \partial_\eta$) with a symbol-type $O(\eta^{-2})$.

Proof. The η^{-3} decay in (A.1) is due to (3.9). Otherwise, the lemma is an immediate consequence of Lemma 4. \square

Now for the transformation of the equation with V, V_0 as in Theorem 1. To motivate our way of obtaining the Bessel equation as an approximating equation, consider the model operator

$$P(x, \hbar D) := -\hbar^2 \partial_x^2 + \langle x \rangle^{-2}$$

It is tempting to introduce the Bessel operator

$$P_0(x, \hbar D) := -\hbar^2 \partial_x^2 + x^{-2}$$

which should be a good approximation for large x . The problem here is that even though the error decays like x^{-4} it is not small compared to \hbar unless $x > \hbar^{-\frac{1}{4}}$. Since we need to be able to send \hbar and E to zero *independently*, such an approximation is useless for the case where E is small but fixed and $\hbar \rightarrow 0$. The idea behind our reduction to the Bessel equation is essentially to let $\langle x \rangle$ be a new independent variable. The reader will easily see that this is precisely what Lemma 4 does (in addition, we scale out E and fix the turning point to lie at 1).

Lemma 12. *For any $0 < E < E_0$ the following holds: $f(x)$ is a smooth solution of*

$$-\hbar^2 f''(x) + V(x)f(x) = Ef(x) \quad \text{on } x > E^{-\frac{1}{2}}$$

iff $\phi(\xi) = \phi(\xi, E) := \Omega(\xi, E)^{\frac{1}{2}} f(E^{-\frac{1}{2}}y(\xi, E))$, with Ω as in (A.2), is a smooth solution of

$$(A.3) \quad -\hbar^2 \phi''(\xi) + [\xi^{-2}(1 - \hbar^2/4) - 1]\phi(\xi) = \hbar^2 W_0(\xi, E)\phi(\xi) \quad \text{on } \xi > \xi_1(E)$$

with a potential W_0 satisfying

$$(A.4) \quad |\partial_E^k \partial_\xi^\ell W_0(\xi, E)| \leq C_{k,\ell} E^{-k} \xi^{-3-\ell}$$

for all $k, \ell \geq 0$.

Proof. Under the change of variables $g(y) = f(E^{-\frac{1}{2}}y)$ the following equations are equivalent, with $V_1(x) = -\frac{1}{4}\langle x \rangle^{-2}$:

$$-\hbar^2 f''(x) + (V_0(x) + \hbar^2 V_1(x))f(x) = Ef(x)$$

$$-\hbar^2 g''(y) + (E^{-1}V_0(E^{-\frac{1}{2}}y) - 1)g(y) = -\hbar^2 E^{-1}V_1(E^{-\frac{1}{2}}y)g(y)$$

Now let $\xi = \xi(y, E)$ be as in Lemma 4 and set $\psi(\xi) = g(y(\xi, E))$, or equivalently, $\psi(\xi(y, E)) = g(y)$. Then, with μ as in Lemma 11,

$$(A.5) \quad \begin{aligned} & -\hbar^2[\psi''(\xi) + (\partial_\xi y(\xi, E))^2(\partial_{yy}\xi)(y(\xi, E), E)\psi'(\xi)] + (\xi^{-2} - 1)\psi(\xi) \\ & = -\hbar^2[\psi''(\xi) + \mu(\xi, E)\psi'(\xi)] + (\xi^{-2} - 1)\psi(\xi) \\ & = -\hbar^2(\partial_\xi y(\xi, E))^2 E^{-1}V_1(E^{-\frac{1}{2}}y(\xi, E))\psi(\xi) \end{aligned}$$

Let Ω be as in Lemma 11. In view of (A.5), $\phi := \Omega^{\frac{1}{2}}\psi$ satisfies the equation

$$(A.6) \quad -\hbar^2 \phi''(\xi) + (\xi^{-2} - 1)\phi(\xi) = \hbar^2 W(\xi, E)\phi(\xi)$$

with

$$W(\xi, E) := -(\partial_\xi y(\xi, E))^2 E^{-1}V_1(E^{-\frac{1}{2}}y(\xi, E)) - \frac{1}{2} \frac{\Omega''(\xi, E)}{\Omega(\xi, E)} + \frac{1}{4} \left(\frac{\Omega'(\xi, E)}{\Omega(\xi, E)} \right)^2$$

The second part here involving Ω decays like ξ^{-4} , whereas the first only decays like ξ^{-2} . We need to extract this leading order decay: the asymptotic expansion

$$V_1(\xi) = -\frac{1}{4\xi^2} + O(\xi^{-3}) \quad \text{as } \xi \rightarrow \infty$$

and Lemma 4 imply that

$$(\partial_\xi y(\xi, E))^2 E^{-1}V_1(E^{-\frac{1}{2}}y(\xi, E)) = -\frac{1}{4\xi^2} + \xi^{-3}V_r(\xi, E)$$

where

$$|\partial_E^k \partial_\xi^\ell V_r(\xi, E)| \leq C_{k,\ell} E^{-k} \xi^{-\ell}$$

In view of (A.6), this yields equation (A.3) and we are done. \square

A fundamental system $\{\phi_{j,n}^{(0)}\}_{j=1}^2$ of the homogeneous form of (A.3), i.e.,

$$(A.7) \quad -\hbar^2 \phi''(\xi) + [\xi^{-2}(1 - \hbar^2/4) - 1]\phi(\xi) = 0$$

is given in terms of Hankel functions:

$$\phi_{j,n}^{(0)}(\xi) = \xi^{\frac{1}{2}} H_n^{(j)}(n\xi), \quad j = 1, 2, \quad n := \hbar^{-1}$$

or, equivalently, by the Bessel functions

$$\tilde{\phi}_{1,n}^{(0)}(\xi) = \xi^{\frac{1}{2}} J_n(n\xi), \quad \tilde{\phi}_{2,n}^{(0)}(\xi) = \xi^{\frac{1}{2}} Y_n(n\xi)$$

with Wronskian

$$W(\tilde{\phi}_{1,n}^{(0)}, \tilde{\phi}_{2,n}^{(0)}) = \frac{2}{\pi}, \quad W(\phi_{1,n}^{(0)}, \phi_{2,n}^{(0)}) = \frac{4i}{\pi}$$

Hence, the forward Green function of (A.7) is

$$\begin{aligned} G_n(\xi, \xi') &:= \frac{\pi}{4i} [\phi_{1,n}^{(0)}(\xi) \phi_{2,n}^{(0)}(\xi') - \phi_{1,n}^{(0)}(\xi') \phi_{2,n}^{(0)}(\xi)] \chi_{[\xi < \xi']} \\ &= \frac{\pi}{2} [\tilde{\phi}_{1,n}^{(0)}(\xi) \tilde{\phi}_{2,n}^{(0)}(\xi') - \tilde{\phi}_{1,n}^{(0)}(\xi') \tilde{\phi}_{2,n}^{(0)}(\xi)] \chi_{[\xi < \xi']} \\ &= \frac{\pi}{2} (\xi \xi')^{\frac{1}{2}} [J_n(n\xi) Y_n(n\xi') - J_n(n\xi') Y_n(n\xi)] \chi_{[\xi < \xi']} \end{aligned}$$

and thus a basis $\{\phi_{j,n}\}_{j=1}^2$ of (A.3) is given by the Volterra equation

$$(A.8) \quad \phi_{j,n}(\xi) = \phi_{j,n}^{(0)}(\xi) + \int_{\xi}^{\infty} G_n(\xi, \xi') W_0(\xi', E) \phi_{j,n}(\xi') d\xi'$$

that one now needs to solve. This of course requires a thorough understanding of the behavior of $J_n(n\xi)$ and $Y_n(n\xi)$ for large n on intervals of the form $\xi > \xi_0 > 0$ where $0 < \xi_0 \ll 1$ is fixed, see [1] and [19]. We leave it to the interested reader to pursue this direction.

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